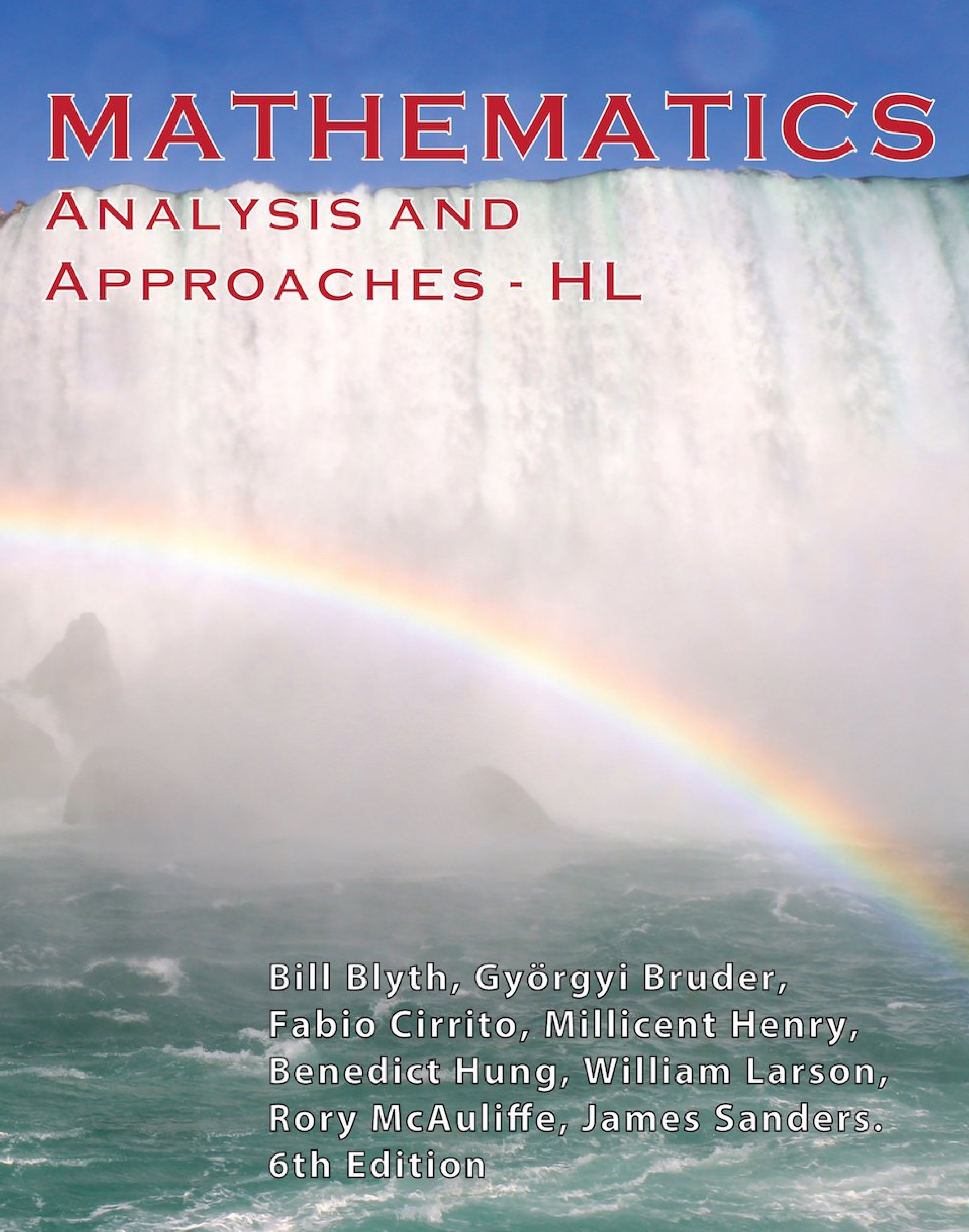


# MATHEMATICS

## Analysis and Approaches - HL

Bill Blyth, Györgyi Bruder,
Fabio Cirrito, Millicent Henry,
Benedict Hung, William Larson,
Rory McAuliffe, James Sanders.

6th Edition

For use with the I.B. Diploma Programme

# MATHEMATICS HL
# ANALYSIS AND APPROACHES

Bill Blyth, Györgyi Bruder,
Fabio Cirrito, Millicent Henry,
Benedict Hung, William Larson,
Rory McAuliffe, James Sanders.

**6th Edition**

www.ibid.com.au

First published in 2019 by IBID Press, Victoria

Library Catalogue:

Bruder, Cirrito, Henry, Hung, Larson, McAuliffe, Sanders

1. Mathematics

2. International Baccalaureate.

Series Title: International Baccalaureate in Detail

ISBN-978-1-921784-82-8



IBID Press expresses its thanks to the International Baccalaureate Organization for permission to reproduce its intellectual property.

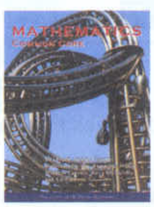
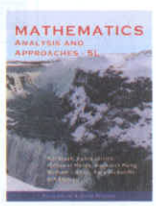
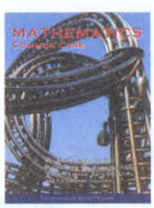
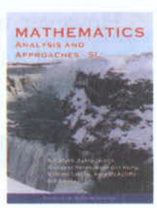
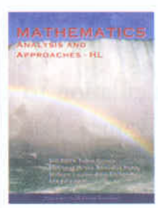
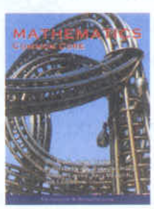
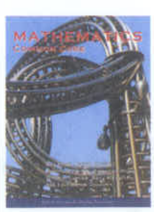
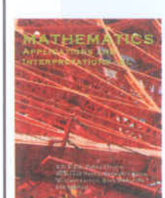
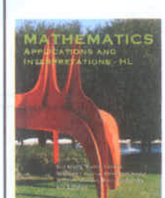
Cover design by Key-Strokes.

Published by IBID Press, www.ibid.com.au

Printed by Red Planet

## Using the 6th Editions of IBID Press Mathematics Texts

This series of texts has been written for the IB Courses Mathematics: Analysis and Approaches and Mathematics: Applications and Interpretations that start teaching in August 2019.

| Course Studied | Common Core | Mathematics: Analysis and Approaches (SL) | Mathematics: Analysis and Approaches (HL) | Mathematics: Applications and Interpretations (SL) | Mathematics: Applications and Interpretations (HL) | Discounted Package |
|---|---|---|---|---|---|---|
| Mathematics: Analysis and Approaches (SL) |  |  | | | | Pure(SL) |
| Mathematics: Analysis and Approaches (HL) |  |  |  | | | Pure(HL) |
| Mathematics: Applications and Interpretations (SL) |  | | |  | | Applied(SL) |
| Mathematics: Applications and Interpretations (HL) |  | | |  |  | Applied(HL) |

# PREFACE

This text for the Mathematics: Analysis and Approaches course has been prepared to closely align with the current course.

It has concise explanations, clear diagrams and calculator references.

Appropriate, graded exercises are provided throughout.

Also relating to International Perspectives and the Theory of Knowledge, it provides more than just the basics. It is an essential resource for those teachers and students who are looking for a reliable guide for their SL course.

This is a re-worked and revised edition of the Standard Level text first published by IBID Press in 1997.

2nd Edition published in 1999

3rd Edition published in 2004

4th Edition published 2012

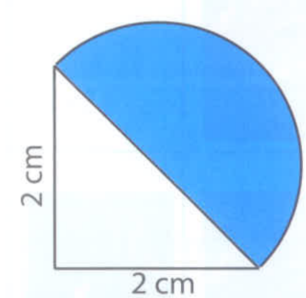
5th Edition published in 2018

6th Edition published in 2019

## Rounding

When carrying an answer from one part of a calculation to a subsequent part, it is best to use unrounded values. For example:

A semicircle is constructed on the hypotenuse of a 2 cm right angled triangle. Find its area correct to 4 significant figures.




Stage 1:

Calculate the length of the hypotenuse using the theorem of Pythagoras:

$\sqrt{2^2+2^2} \approx 2.828 (4 \text{ s.f.})$

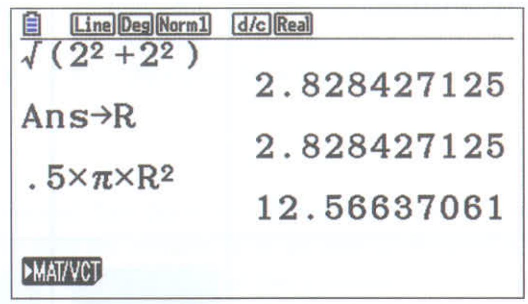
Stage 2

Find the area of the circle and halve it:

Area $= \frac{1}{2}\pi r^2 \approx \frac{1}{2}\pi \times 2.828^2 \approx 12.56$ cm (4 s.f.)

In this text, we will show calculations in this way as, we believe, students will be able to follow our explanations more easily if we do this.

However, if using a calculator, the best procedure is to use the memory to store a full accuracy version of the radius (second line).




Rounding this gives the answer 12.57 cm$^2$ (4 s.f.)

Note that the two answers are different. We understand that both answers are usually marked correct in examinations. However, we suggest that using the memory and the calculator value of π (not 3.14) is the better method.

## Calculators

Students who are thoroughly familiar with the capabilities of their model of calculator place themselves at a considerable advantage over students who are not.

In preparing a text such as this, we cannot provide an exhaustive account of every place in which a calculator can help. Or, for that matter, an explanation of how each model works!

This text uses examples from Texas Instruments and Casio graphic calculators.

The manufacturers all provide extensive 'manuals'. These can be intimidating.

We suggest that a good strategy is to take each topic and, as you are learning it, take some time to discover your model's capability in that topic.

For example, Section 1.3 deals with counting principles. It is highly likely your calculator will be very helpful here. A good strategy can be to 'Google' or 'Bing' your model plus the topic.

There are now a number of training videos available on YouTube.

## Answers

Answers to the Exercises are available as a free download from the publisher's website:

**www.ibid.com.au**

Also, there are QR codes embedded in the text that link directly to these.

### Online Errata

### Supplementary Material

# TABLE OF CONTENTS


## A: NUMBER AND ALGEBRA

A.5 Counting Principles 2

A.6 Partial Fractions 15

A.7 Complex Numbers 23

A.8 Proof 47

A.9 Systems of Linear 57

## B: FUNCTIONS

B.5 Factor and Remainder Theorem 66

B.6 Rational Functions 77

B.7 Further Functions 85

B.8 Modulus Function and Solving Inequalties 93

## C: TRIGONOMETRY AND GEOMETRY

C.8 Reciprocal and Inverse Trigonometric Functions 104

C.9 Further Identities 115

C.10 Trigonometric Functions 119

C.11 Vectors 125

## D: STATISTICS AND PROBABILITY

D.7 Bayes' Theorem 180

D.8 Further Probability 185

## E: CALCULUS

E.7 Continuity and Differentiability 200

E.8 Further Limits 209

E.9 Implicit Differentiation 213

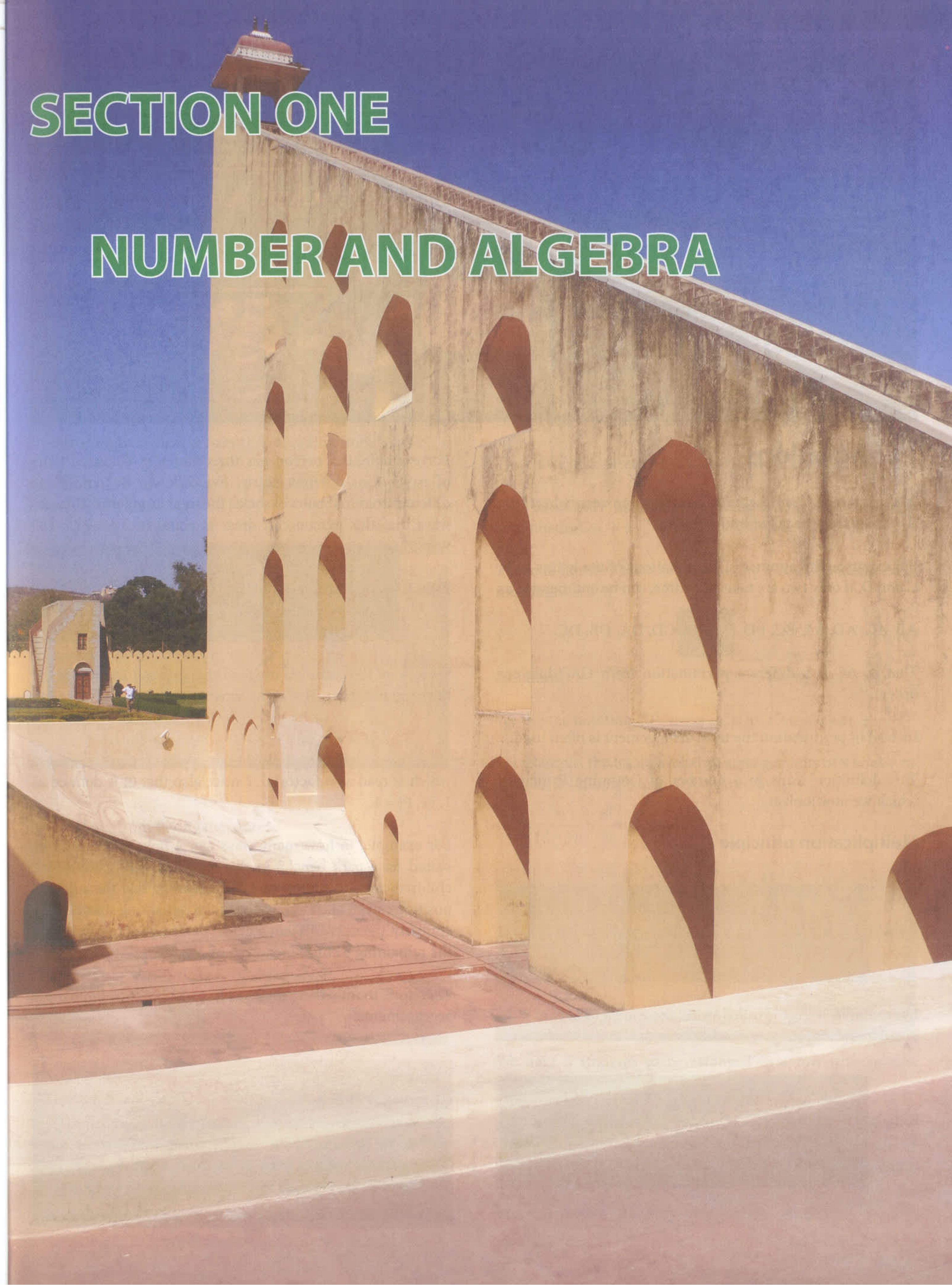
E.10 Integration Methods 225

E.11 Differential Equations 247

Index 265

# SECTION ONE

# NUMBER AND ALGEBRA



# A.5 Counting Principles

AHL 1.10

## Permutations

Permutations represents a counting process where the order must be taken into account.

For example, the number of permutations of the letters A, B, C and D, if only two are taken at a time, can be enumerated as

AB, AC, AD, BA, BC, BD, CA, CB, CD, DA, DB, DC

That is, AC is a different permutation from CA (different order).

Instead of permutation the term arrangement is often used.

This definition leads to a number of Counting Principles which we now look at.

### Multiplication principle

**Rule 1:** If any one of $n$ different mutually exclusive and exhaustive events can occur on each of $k$ trials, the number of possible outcomes is equal to $n^k$.

For example, if a die is rolled twice, there are a total of $6^2 = 36$ possible outcomes.

**Rule 2:** If there are $n_1$ events on the first trial, $n_2$ events on the second trial, and so on, and finally, $n_k$ events on the $k$th trial, then the number of possible outcomes is equal to $n_1 \times n_2 \times ... \times n^k$.

For example, if a person has three different coloured pairs of pants, four different shirts, five different ties and three different coloured pairs of socks, the total number of different ways that this person can dress is equal to $3 \times 4 \times 5 \times 3 = 180$ ways.

**Rule 3:** The total number of ways that $n$ different objects can be arranged in order is equal to:

$$n \times (n-1) \times (n-2) \times \ldots 3 \times 2 \times 1.$$

Because of the common usage of this expression, we use the **factorial notation**. That is, we write:

$$n! = n \times (n-1) \times (n-2) \times \ldots 3 \times 2 \times 1.$$

which is read as ***n* factorial**. Notice also that 0! is defined as 1, i.e. $0! = 1$.

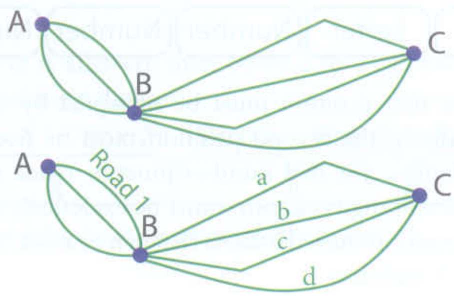
For example, in how many ways can 4 boys and 3 girls be seated on a park bench? In this case any one of the seven children can be seated at one end, meaning that the adjacent position can be filled by any one of the remaining six children, similarly, the next adjacent seat can be occupied by any one of the remaining 5 children, and so on . . .

Therefore, in total there are $7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5040$ possible arrangements.

**Example A.5.1**

John wishes to get from town A to town C via town B. There are three roads connecting town A to town B and 4 roads connecting town B to town C. In how many different ways can John get from town A to town C?

We start by visualizing this situation:




Consider the case where John uses Road 1 first.

The possibilities are:.

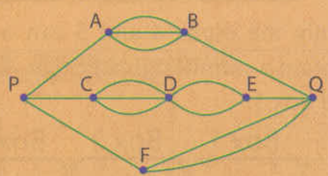
Road 1 then a, Road 1 then b, Road 1 then c, Road 1 then d.

That is, there are 4 possible routes. Then, for each possible road from A to B there are another 4 leading from B to C.

All in all, there are 4+4+4 = 12 different ways John can get from A to C via B.

### Example A.5.2

Using the following street network, in how many different ways can a person get from point P to point Q if they can only move from left to right?




In travelling from P to Q there are:

$3 = 1 \times 3 \times 1$ paths (along P to A to B to Q)

$6 = 1 \times 3 \times 2 \times 1$ paths (along P to C to D to E to Q)

$2 = 1 \times 2$ paths (along P to F to Q)

In total there are $3 + 6 + 2 = 11$ paths

### Example A.5.3

A golfer has 3 drivers, 4 tees and 5 golf balls. In how many ways can the golfer take his first stroke?

The golfer has 3 possible drivers to use and so the first task can be carried out in 3 ways.

The golfer has 4 possible tees to use and so the second task can be carried out in 4 ways.

The golfer has 5 golf balls to use and so the third task can be carried out in 5 ways.

Using the multiplication principle, there are a total of $3 \times 4 \times 5 = 60$ ways to take the first stroke.

## Permutations

A permutation is an arrangement in which both the items chosen and the order in which they are chosen matter.

Thus if we pick three letters from the word PENCIL:

PEN and EPN are different permutations but the same combination.

The total number of ways of arranging $n$ objects, taking $r$ at a time is given by:

$$\frac{n!}{(n-r)!}$$

**Notation:**

We use the notation ${}^nP_r$ (read as "n–p–r") to denote $\frac{n!}{(n-r)!}$

For example, the total number of arrangements of 8 books on a bookshelf if only 5 are used is given by:

$${}^8P_5 = \frac{8!}{(8-5)!} = \frac{8!}{3!} = 6720$$

### Example A.5.4

In how many ways can 5 students be arranged in a row:

a using three students at a time?

b using 5 students at a time?

We have 5 students to be arranged in a row with certain constraints.

a The constraint is that we can only use 3 students at a time. In other words, we want the number of arrangements (permutations) of 5 objects taken 3 at a time.

Therefore: $n = 5, r = 3,$

The number of arrangements is: $^5P_3 = \frac{5!}{(5-3)!} = \frac{120}{2} = 60$

b This time we want the number of arrangements of 5 boys taking all 5 at a time. Therefore: $n = 5, r = 5,$

The number of arrangements is: $^5P_5 = \frac{5!}{(5-5)!} = \frac{120}{0!} = 120$

## Box method

Problems like Example A.5.3 can be solved using a method known as "the box method". In that particular example, part (a) can be considered as filling three boxes (with only one object per box) using 5 objects:

| Box 1 | Box 2 | Box 3 |
|---|---|---|
| | | |

The first box can be filled in 5 different ways (as there are 5 possibilities available). Therefore we 'place 5' in box 1:

| Box 1 | Box 2 | Box 3 |
|---|---|---|
| 5 | | |

Now, as we have used up one of the objects (occupying box 1), we have 4 objects left that can be used to fill the second box. So, we 'place 4' in box 2:

| Box 1 | Box 2 | Box 3 |
|---|---|---|
| 5 | 4 | |

At this stage we are left with three objects (as two of them have been used). This means that there are 3 possible ways in which the third box can be filled. So, we 'place 3' in box 3:

| Box 1 | Box 2 | Box 3 |
|---|---|---|
| 5 | 4 | 3 |

This is equivalent to saying, that we can carry out the first task in 5 different ways, the second task in 4 different ways and the third task in 3 different ways. Therefore, using the multiplication principle we have that the total number of arrangements is 5 × 4 × 3 = 60 - the same answer as the permutations method.

### Example A.5.5

Vehicle licence plates consist of two letters from a 26-letter alphabet, followed by a three-digit number whose first digit cannot be zero. How many different licence plates can there be?

We have a situation where there are five positions to be filled:

| Letter | Letter | Number | Number | Number |
|---|---|---|---|---|

That is, the first position must be occupied by one of 26 letters, similarly, the second position must be occupied by one of 26 letters. The first number must be made up of one of nine different digits (as zero must be excluded), whilst the other two positions have 10 digits that can be used. Therefore, using Rule 2, we have:

Total number of arrangements $=26\times26\times9\times10\times10 = 608\ 400$.

### Example A.5.6

How many 5-digit numbers greater than 40,000 can be formed from 0, 1, 2, 3, 4, and 5 if:

a there is no repetition of digits allowed?

b repetition of digits is allowed?

a Consider the five boxes:

| Box 1 | Box 2 | Box 3 | Box 4 | Box 5 |
|---|---|---|---|---|
| | | | | |

Only the digits 4 and 5 can occupy the first box (so as to obtain a number greater than 40,000). So there are 2 ways to fill box 1:

| Box 1 | Box 2 | Box 3 | Box 4 | Box 5 |
|---|---|---|---|---|
| 2 | | | | |

Box 2 can now be filled using any of the remaining 5 digits. So, there are 5 ways of filling box 2:

| Box 1 | Box 2 | Box 3 | Box 4 | Box 5 |
|---|---|---|---|---|
| 2 | 5 | | | |

We now have 4 digits left to be used. So, there are 4 ways of filling box 3:

| Box 1 | Box 2 | Box 3 | Box 4 | Box 5 |
|---|---|---|---|---|
| 2 | 5 | 4 | | |

Continuing in this manner we have:

| Box 1 | Box 2 | Box 3 | Box 4 | Box 5 |
|---|---|---|---|---|
| 2 | 5 | 4 | 3 | 2 |

Then, using the multiplication principle we have:
$2\times5\times4\times3\times2 = 240$ arrangements.

Otherwise, we could have relied on permutations and obtained:

$2\times{}^5P_4 = 2\times120 = 240$ arrangements.

b As in part a, only the digits 4 and 5 can occupy the first box.

If repetition is allowed, then boxes 2 to 5 can each be filled using any of the 6 digits:

| Box 1 | Box 2 | Box 3 | Box 4 | Box 5 |
|---|---|---|---|---|
| 2 | 6 | 6 | 6 | 6 |

Using the multiplication principle there are:

$2\times6\times6\times6\times6 = 2592$ arrangements.

However, one of these arrangements will also include the number 40 000. Therefore, the number of 5 digit numbers greater than 40,000 (when repetition is allowed) is given by $2592 - 1 = 2591$.

**Example A.5.7**

Find $n$ if: ${}^{n}P_3 = 60$

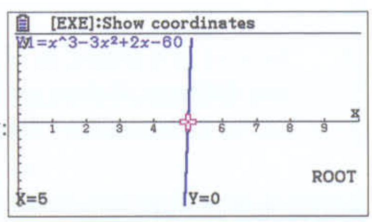
$${}^{n}P_3 = 60 \Leftrightarrow \frac{n!}{(n-3)!} = 60$$

$$\Leftrightarrow \frac{n(n-1)(n-2)(n-3)!}{(n-3)!} = 60$$

$$n(n-1)(n-2)(n-3) = 60$$

$$n^3 - 3n^2 + 2n - 60 = 0$$

This is best solved graphically:




$n = 5$

**Example A.5.8**

How many different arrangements of the letters of the word HIPPOPOTAMUS are there?

The word 'HIPPOPOTAMUS' is made up of 12 letters, unfortunately, they are not all different! This means that, although we can swap the three Ps with each other, the word will remain the same.

Now, the total number of times we can rearrange the Ps (and not alter the word) is 3! = 6 times (as there are three Ps). Therefore, if we 'blindly' use Rule 2, we will have increased the number of arrangements 6 fold.

Therefore, we will need to divide the total number of ways of arranging 12 objects by 6.

That is, $\frac{12!}{3!} = 79833600$.

However, we also have 2 Os, and so, the same argument holds. So that in fact, we now have a total of:

$$\frac{12!}{3!\times2!} = 39916800 \text{ arrangements}$$

This example is a special case of permutations with repetitions:

**Rule:** The number of permutations of $n$ objects of which $n_1$ are identical, $n_2$ are identical, . . . , $n_k$ are identical is given by:

$$\frac{n!}{n_1!\times n_2!\times\ldots\times n_k!}$$

## Exercise A.5.1

1. A, B and C are three towns. There are 5 roads linking towns A and B and 3 roads linking towns B and C. How many different paths are there from town A to town C via town B?

2. In how many ways can 5 letters be mailed if there are:

   a 2 mail boxes available?

   b 4 mail boxes available?

3. There are 4 letters to be placed in 4 letter boxes. In how many ways can these letters be mailed if:

   a only one letter per box is allowed?

   b there are no restrictions on the number of letters per box?

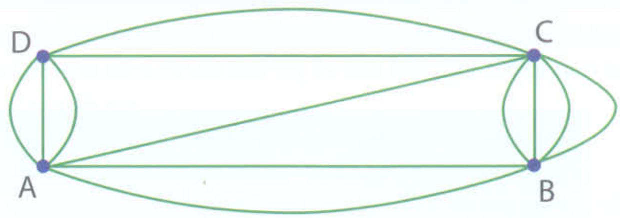
4. Consider the cubic polynomial:

   $$p(x) = ax^3 + bx^2 - 5x + c$$

   a If the coefficients, $a$, $b$ and $c$ come from the set $\{-3, -1, 1, 3\}$, find the number of possible cubics if no repetitions are allowed.

b Find the number of cubics if the coefficients now come from {-3, -1, 0, 1, 3} (again without repetitions).

5. The diagram shows the possible routes linking towns A, B, C and D.




A person leaves town A for town C. How many different routes can be taken if the person is always heading towards town C?

6. In how many different ways can Susan get dressed if she has 3 skirts, 5 blouses, 6 pairs of socks and 3 pairs of shoes to chose from?

7. In how many different ways can 5 different books be arranged on a shelf?

8. In how many ways can 8 different boxes be arranged taking 3 at a time?

9 How many different signals can be formed using 3 flags from 5 different flags?

10 Three Italian, two chemistry and four physics books are to be arranged on a shelf.

In how many ways can this be done if:

a there are no restrictions?

b the chemistry books must remain together?

c the books must stay together by subject?

11. Find $n$ if: ${}^{n}P_{5} = 380$ .

12. Five boys and six girls, which include a brother-sister pair, are to be arranged in a straight line. Find the number of possible arrangements if:

a there are no restrictions.

b the tallest must be at one end and the shortest at the other end.

c the brother and sister must be: i together ii separated.

13. In how many ways can the letters of the word Mississippi be arranged?

14. In how many ways can three yellow balls, three red balls and four orange balls be arranged in a row if the balls are identical in every way other than their colour?

15. In a set of 8 letters, m of them are the same and the rest different. If there are 1680 possible arrangements of these 8 letters, how many of them are the same?

## Combinations

On the other hand, combinations represent a counting process where the order has no importance. For example, the number of combinations of the letters A, B, C and D, if only two are taken at a time, can be enumerated as:

AB, AC, AD, BC,BD, CD,

That is, the combination of the letters A and B, whether written as AB or BA, is considered as being the same.

Instead of combination the term selection is often used.

The total number of ways of selecting $n$ objects, taking $r$ at a time is given by:

$$\frac{n!}{(n-r)!r!}$$

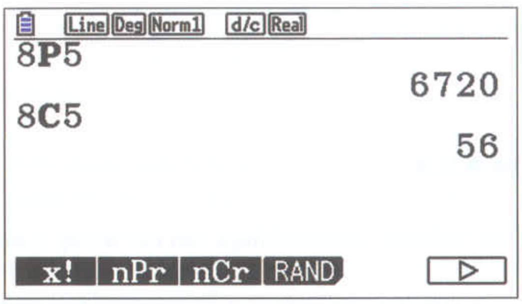
**Notation:**

We use the notation $^nC_r$ (read as "n-c-r") or $\begin{pmatrix} n \\ r \end{pmatrix}$ to denote $\frac{n!}{(n-r)!r!}$.

For example, in how many ways can 5 books be selected from 8 different books? In this instance, we are talking about selections and therefore, we are looking at combinations. Therefore we have, the selection of 8 books taking 5 at a time is equal to:

$$\begin{pmatrix} 8 \\ 5 \end{pmatrix} = \frac{8!}{(8-5)!5!} = \frac{8!}{3!5!} = 56$$

Graphic calculators mostly have a 'probability menu' which you should locate.




**Example A.5.9**

A sports committee at the local hospital consists of 5 members. A new committee is to be elected, of which 3 members must be women and 2 members must be men. How many different committees can be formed if there were originally 5 women and 4 men to select from?

First we look at the number of ways we can select the women members:

We have to select 3 from a possible 5, therefore, this can be done in $^5C_3 = 10$ ways.

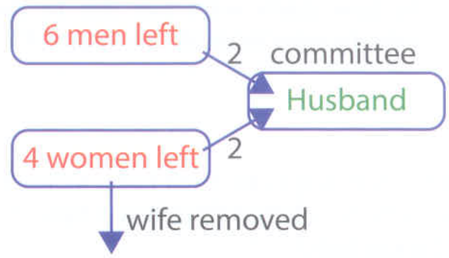
Similarly, the men can be selected in $^4C_2 = 6$ ways.

Using Rule 2, we have that the total number of possible committees $= {}^5C_3 \times {}^4C = 60$ ways.

**Example A.5.10**

A committee of 3 men and 2 women is to be chosen from 7 men and 5 women. Within the 12 people there is a husband and wife. In how many ways can the committee be chosen if it must contain either the wife or the husband but not both?

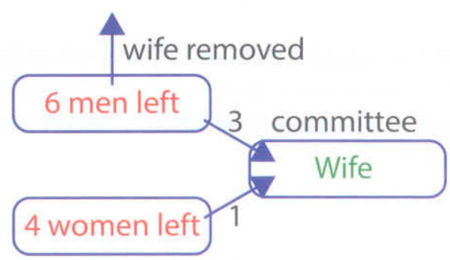
Case 1: Husband included




If the husband is included, the wife must be removed (so that she cannot be included). We then have to select 2 more men from the remaining 6 men and 2 women from the remaining 4 women.

This is done in $^6C_2 \times {}^4C_2 = 90$ ways.

Case 2: Wife included




If the wife is included, the husband must be removed. We then have to select 3 men from the remaining 6 men and 1 woman from the remaining 4 women.

This is done in: $^6C_3 \times {}^4C_1 = 80$ ways.

Therefore there are a total of:

$^6C_2 \times {}^4C_2 + {}^6C_3 \times {}^4C_1 = 90 + 80 = 170$ possible committees.

### Exercise A.5.2

1. In how many ways can 5 basketball players be selected from 12 players?

2. A tennis club has 20 members.

   a In how many ways can a committee of 3 be selected.

   b In how many ways can this be done if the captain must be on the committee?

3. In how many ways can 3 red balls, 4 blue balls and 5 white balls be selected from 5 red balls, 5 blue balls and 7 white balls?

4. In how many ways can 8 objects be divided into 2 groups of 4 objects?

5. A cricket training squad consists of 4 bowlers, 8 batsmen, 2 wicket keepers and 4 fielders.

   From this squad a team of 11 players is to be selected. In how many ways can this be done if the team must consist of 3 bowlers, 5 batsmen, 1 wicket keeper and 2 fielders?

6. A class consists of 12 boys of whom 5 are prefects. How any committees of 8 can be formed if the committee is to have:

   a 3 prefects?

   b at least 3 prefects?

7. In how many ways can 3 boys and 2 girls be arranged in a row if a selection is made from 6 boys and 5 girls?

8. If $\binom{n}{3}=56$ show that $n^3-3n^2+2n-336=0$.

   Hence find $n$.

9. In how many ways can a jury of 12 be selected from 9 men and 6 women so that there are at least 6 men and no more than 4 women on the jury.

10. Show that $\binom{n+1}{3}-\binom{n-1}{3}=(n-1)^2$.

    Hence find $n$ if: $\binom{n+1}{3}-\binom{n-1}{3}=16$

### Exercise A.5.3

1. Five different coloured flags can be run up a mast.

   a How many different signals can be produced if all five flags are used?

   b How many different signals can be produced if any number of flags is used?

2. In how many different ways can 7 books be arranged in a row?

3. In how many different ways can 3 boys and 4 girls be seated in a row?

   In how many ways can this be done if:

   a no two girls are sitting next to each other?

   b the ends are occupied by girls?

4. In how many different ways can 7 books be arranged in a row if:

   a three specified books must be together?

   b two specified books must occupy the ends?

5. A school council consists of 12 members, 6 of whom are parents and 2 are students, the Principal and the remainder are teachers. The school captain and vice-captain must be on the council. If there are 10 parents and 8 teachers nominated for positions on the school council, how many different committees can there be?

6. A committee of 5 men and 5 women is to be selected from 9 men and 8 women.

a How many possible committees can be formed?

b Amongst the 17 people, there is a married couple. If the couple cannot serve together, how many committees could there be?

7. A sports team consists of 5 bowlers (or pitchers), 9 batsmen and 2 keepers (or back-stops).

How many different teams of 11 players can be chosen from the above squad if the team consists of:

a 4 bowlers (pitchers), 6 batsmen and 1 keeper (back-stop)?

b 6 batsmen (batters) and at least 1 keeper (back-stop)?

8. Twenty people are to greet each other by shaking hands. How many handshakes are there?

9. How many arrangements of the letters of the word "MARRIAGE" are possible?

10. How many arrangements of the letters of the word "COMMISSION" are possible?

11. A committee of 4 is to be selected from 7 men and 6 women. In how many ways can this be done if:

a there are no restrictions?

b there must be an equal number of men and women on the committee?

c there must be at least one member of each sex on the committee?

12. Prove that:

a $\binom{n}{r}+\binom{n}{r+1}=\binom{n+1}{r+1}$

b ${}^{n+1}P_r={}^{n}P_r+r\times{}^{n}P_{r-1}$

13. A circle has $n$ points on its circumference. How many chords joining pairs of points can be drawn?

14. A circle has $n$ points on its circumference. What is the maximum number of points of intersection of chords inside the circle?

15.

a Show that: $2^n=\sum_{r=1}^{n}\binom{n}{r}$

b In how many ways can 8 boys be divided into two unequal sets?

16. Whilst at the library, Patrick decides to select 5 books from a group of 10. In how many different ways can Patrick make the selection?

17. A fish tank contains 5 gold-coloured tropical fish and 8 black-coloured tropical fish.

a In how many ways can five fish be selected?

b If a total of 5 fish have been selected from the tank, how many of these contain two gold fish?

18. In how many ways can 4 people be accommodated if there are 4 rooms available?

19. A car can hold 3 people in the front seat and 4 in the back seat. In how many ways can 7 people be seated in the car if John and Samantha must sit in the back seat and there is only one driver?

20. In how many ways can six men and two boys be arranged in a row if:

    a the two boys are together?

    b the two boys are not together?

    c there are at least three men separating the boys?

21. In how many ways can the letters of the word "TOGETHER" be arranged? In how many of these arrangements are all the vowels together?

22. In how many ways can 4 women and 3 men be arranged in a row, if there are 8 women and 5 men to select from?

23. In how many ways can 4 women and 3 men be arranged in a circle? In how many ways can this be done if the tallest woman and shortest man must be next to each other?

24. In how many ways can 5 maths books, 4 physics books and 3 biology books be arranged on a shelf if subjects are kept together?

25. How many even numbers of 4 digits can be formed using 5, 6, 7, 8 if:

    a no figure is repeated?

    b repetition is allowed?

26. Five men and five women are to be seated around a circular table. In how many ways can this be done if the men and women alternate?

27. A class of 20 students contains 5 student representatives. A committee of 8 is to be formed. How many different committees can be formed if there are:

    a only 3 student representatives?

    b at least 3 student representatives?

28. How many possible juries of 12 can be selected from 12 women and 8 men so that there are at least 5 men and not more than 7 women?

29. In how many ways can 6 people be seated around a table if 2 friends are always:

    a together?

    b separated?

## Binomial Theorem

We have met the Binomial Theorem in the context of multiplying out brackets:

$$(a+b)^n = \begin{bmatrix} n \\ 0 \end{bmatrix} a^0 b^n + \begin{bmatrix} n \\ 1 \end{bmatrix} a^1 b^{n-1} + \begin{bmatrix} n \\ 2 \end{bmatrix} a^2 b^{n-2} + \ldots + \begin{bmatrix} n \\ n \end{bmatrix} a^n b^0$$

where $\begin{bmatrix} n \\ r \end{bmatrix} = {}^nC_r$

There are several ways of calculating the binomial coefficients: Pascal's Triangle and combinatorial numbers being the most common.

Can we extend the Binomial Theorem to cases where the index is not a positive whole number?

The answer is 'yes', without using a different 'pattern' and with some very useful consequences.

**Example A.5.11**

Find the Binomial expansion of $(1+x)^{-1}$

If we replicate the pattern used in other binomial expansions, there are two aspects that we need to consider.

1. Pattern of terms

This means that one of the powers starts at '$n$' and decreases by one with each term and the other begins at zero and increases by 1. As one of the terms is 1, the pattern of terms can be written with the power of 1 decreasing and the power of $x$ increasing. Since 1 to any power is 1, we only need to worry about the powers of $x$: $x, x^2, x^3, x^4, \ldots$

2. Coefficients

Since these are to be combinatorial numbers, it is usual to think of these as being expressible in this way:

For example: ${}^6C_2 = \frac{6!}{(6-2)!2!} = \frac{6!}{4!2!} = \frac{6.5.4.3.2.1}{4.3.2.1 \times 2.1} = \frac{6.5}{2.1}$

In this case, the numerator is a 'terminated factorial' - a factorial that does not run all the way down to one because of the cancellation of terms with those in the denominator. If we think in this way, the binomial coefficients become:

$$ {}^nC_0 = \frac{n(n-1)(n-2)(n-3)\ldots3.2.1}{n(n-1)(n-2)(n-3)\ldots3.2.1 \times 0!} = 1 $$

$$ {}^nC_1 = \frac{n(n-1)(n-2)(n-3)\ldots3.2.1}{(n-1)(n-2)(n-3)\ldots3.2.1 \times 1!} = n $$

$$ {}^nC_2 = \frac{n(n-1)(n-2)(n-3)\ldots3.2.1}{(n-2)(n-3)\ldots3.2.1 \times 2!} = \frac{n(n-1)}{2!} $$

$$ {}^nC_3 = \frac{n(n-1)(n-2)(n-3)\ldots3.2.1}{(n-3)\ldots3.2.1 \times 3!} = \frac{n(n-1)(n-2)}{3!} $$

The terms in red are those that cancel. Notice that a pattern is beginning to emerge. It is a good idea to write out a few more terms for yourself just to check that it continues.

We can now put these two features of the expansion together to get:

$$ (1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots $$

If $n$ is a positive whole number, this series terminates at the moment when the numerator reaches the term:

$n(n-1)(n-2)(n-3)\ldots(n-n)$ which will be zero, as will all subsequent terms. We have a finite expansion of the type dealt with earlier.

If, however, $n$ is negative or fractional, we will miss this zero and the series will be infinite.

This is the case with this example.

The required series is:

$$ (1+x)^{-1} $$
$$ = 1 + (-1)x + \frac{(-1)((-1)-1)}{2!}x^2 + \frac{(-1)((-1)-1)((-1)-2)}{3!}x^3 + \ldots $$
$$ = 1 + (-1)x + \frac{(-1)(-2)}{2!}x^2 + \frac{(-1)(-2)(-3)}{3!}x^3 + \ldots $$
$$ = 1 - x + x^2 - x^3 + \ldots $$

As we have said, this sequence is infinite. Such series converge if $|x| < 1$ and diverge otherwise.

As a final comment, the series we have generated is a geometric series with first term 1 and common ratio $-x$. Using the sum of such series:

$S_\infty = \frac{a}{1-r} = \frac{1}{1-(-x)} = \frac{1}{1+x} = (1+x)^{-1}$ the original expression.

**Example A.5.12**

Use the Binomial expansion of $(1+x)^{-1}$ to find the reciprocal of 1.2 correct to 5 significant figures.

Using a value of $x = 0.2$:

$$ (1+x)^{-1} = 1 - x + x^2 - x^3 + \ldots $$
$$ (1+0.2)^{-1} = 1 - 0.2 + 0.2^2 - 0.2^3 + \ldots $$
$$ = 1 - 0.2 + 0.04 - 0.008 + \ldots $$

The terms are getting smaller and the series is converging, but how many terms should we use to be sure that the answer is correct to 5 significant figures? It appears that the answer is in the region of 0.8 so we should continue until we get a term that is zero to 5 decimal places. It is a good idea to go on place further and then round to the required accuracy.

$$ (1+0.2)^{-1} = 1 - 0.2 + 0.04 - 0.008 + 0.0016 - 0.00032 + \ldots $$
$$ \ldots + 0.000064 - 0.0000128 + 0.00000256 - \ldots $$
$$ \ldots 0.000000512 + \ldots $$
$$ = 0.833333248 $$

The answer can now be safely rounded to 0.83333 (5 s.f.).

**Example A.5.13**

Use the Binomial expansion of $\sqrt{1+x}$ to find the square root of 1.1 correct to 4 significant figures.

Using:

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$$

with $n = ½$.

$$(1+x)^{\frac{1}{2}} = 1 + \tfrac{1}{2}x + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}x^2 + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}x^3 + \ldots$$

$$= 1 + \tfrac{1}{2}x + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2!}x^2 + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}x^3 + \ldots$$

$$= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \frac{7}{256}x^5 - \ldots$$

In order to estimate the value of $\sqrt{1.1}$, the value of $x = 0.1$.

$$\sqrt{1+0.1} = 1 + \frac{1}{2}(0.1) - \frac{1}{8}(0.1)^2 + \frac{1}{16}(0.1)^3 - \frac{5}{128}(0.1)^4 + \ldots$$

$$= 1 + 0.05 - 0.00125 + 0.0000625 - \ldots$$

$$= 1.0488125$$

$$= 1.049 \text{ to 4 s.f.}$$

**Example A.5.14**

Use the Binomial Theorem to find $\frac{1}{\sqrt[3]{1.21}}$ correct to 4 significant figures.

We could use $n = -^1/_3$ and $x = 0.21$.

However, a neater result is found by observing that $1.1^2 = 1.21$ and that since:

$$\frac{1}{\sqrt[3]{1.21}} = \frac{1}{\sqrt[3]{1.1^2}} = \frac{1}{1.1^{\frac{2}{3}}}$$

we can use $n = -^2/_3$ and $x = 0.1$.

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$$

The required expansion is:

$$(1+x)^{-\frac{2}{3}} = 1 + \left(-\frac{2}{3}\right)x + \frac{\left(-\frac{2}{3}\right)\left(\left(-\frac{2}{3}\right)-1\right)}{2!}x^2 + \ldots$$

$$\ldots + \frac{\left(-\frac{2}{3}\right)\left(\left(-\frac{2}{3}\right)-1\right)\left(\left(-\frac{2}{3}\right)-2\right)}{3!}x^3 + \ldots$$

As we do not know how many terms to take, we can evaluate the series so far:

$$(1+0.1)^{-\frac{2}{3}} = 1 - \frac{2}{3}(0.1) + \frac{5}{9}(0.1)^2 - \frac{40}{81}x^3 + \ldots$$

$$= 1 - 0.066666 + 0.00555555 - 0.0004938\ldots$$

$$= 0.9383958$$

Is this far enough? The only way of deciding this is to compute the next term:

$$\frac{\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)\left(-\frac{8}{3}\right)\left(-\frac{11}{3}\right)}{4!}x^4 = +\frac{110}{243}x^4$$

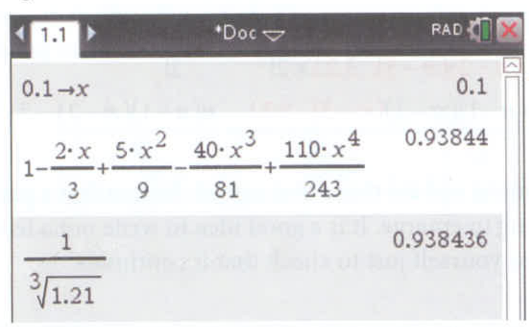
With $x = 0.1$, this is:

$$+\frac{110}{243}(0.1)^4 \approx 0.0000453$$

If this is added to the total, we now get 0.93844106. As the next term is negative and smaller than the last one, we can now quote our answer:

$$\frac{1}{\sqrt[3]{1.21}} = 0.9384 \text{ to 4 s.f.}$$

Checking this with a calculator:




**Example A.5.15**

Use the Binomial Theorem to expand $\frac{1}{(1-x)^3}$ up to the term in $x^4$.

Using the expansion:

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$$

$n = -3$ and we replace $x$ with $(-x)$.

$$(1+(-x))^n$$
$$=1+n(-x)+\frac{n(n-1)}{2!}(-x)^2+\frac{n(n-1)(n-2)}{3!}(-x)^3+...$$

The expansion becomes:

$$(1+(-x))^{-3}=1+(-3)(-x)+\frac{(-3)((-3)-1)}{2!}(-x)^2+...$$
$$...+\frac{(-3)((-3)-1)((-3)-2)}{3!}(-x)^3+...$$
$$...+\frac{(-3)((-3)-1)((-3)-2)((-3)-3)}{4!}(-x)^4+...$$
$$=1+3x+\frac{(-3)(-4)}{2!}x^2+\frac{(-3)(-4)(-5)}{3!}\times -x^3+...$$
$$...+\frac{(-3)(-4)(-5)(-6)}{4!}x^4$$
$$=1+3x+6x^2+10x^3+15x^4$$

## Exercise A.5.4

1. Expand $(1+x)^{-2}$ up to the term in $x^4$.

   a Use your expansion to estimate $\frac{1}{1.5^2}$

   b Comment on the level of accuracy of your answer.

2. Expand $\frac{1}{\sqrt{1+x}}$ up to the term in $x^5$.

   a Use your expansion to estimate $\frac{1}{\sqrt{1.01}}$

   b Comment on the level of accuracy of your answer.

3. Find the term in $x^6$ in the expansions of:

   a $(1-x)^{1.1}$

   b $\sqrt{1-x}$

   c $(1+2x)^{-1}$

   d $\sqrt{1-2x}$

4. Expand, using the Binomial Theorem, up to the term in $x^5$, the following:

   a $\frac{1}{\sqrt{1+3x}}$

   b $(1-3x)^{0.7}$

   c $\frac{1}{\sqrt[3]{1+2x}}$

   d $2\sqrt{1+x}$

5. Consider the expression $\frac{3}{(1-x)^2}$

   a Use the Binomial theorem to develop a series expansion.

   b Substitute $x = 0.2$ into the first seven terms of your expansion

   c Use your expansion to approximate $\frac{3}{0.9^2}$

6. Find the term in $x^5$ in the binomial expansion of $4\sqrt{1-2x}$.

7. Consider the expression $4+x$.

   a Write the expression in the form $A(1+Bx)$ where $A$ & $B$ are constants.

   b Use your expression to find a series expansion for $\sqrt{4+x}$.

   c Hence find the square root of 4.1 correct to 5 significant figures.

8. Consider the expression $\frac{1}{(1-5x)^3}$

   a Find the first three terms in the Binomial Expansion of this expression.

   b Find the coefficient of the term in $x^6$.

Your answer to part b suggests that the size of the terms might be growing and the series diverging even if $|x| < 1$. Us a value of x = 0.5 to answer the rest of this question.

c Find the ratio of term 2 to term 1. Are the terms growing in size or decreasing?

d Find the ratio of term 4 to term 3. Are the terms growing in size or decreasing?

e Find the ratio of term 7 to term 6. Are the terms growing in size or decreasing?

f Is this series a viable method of making numerical approximations.

9. Find the first seven terms in the expansion of $\frac{1}{\sqrt[4]{1+x}}$.

a Find the value of $\frac{1}{\sqrt[4]{1.1}}$ to the maximum accuracy permitted by your series.

b Find the absolute error of your estimate from part a.

c Find the percentage error of your estimate from part a.

10. Find the coefficient of the term in $x^4$ in the binomial expansion of $(1-2x)^{-0.1}$.

11. Use a series method to find the value of $\sqrt{2}$ correct to 4 significant figures.

12. Expand both $\sqrt{1-x}$ and $\frac{1}{(1+x)^2}$ as far as the terms in $x^4$.

a Hence expand $\frac{\sqrt{1-x}}{(1+x)^2}$

b Hence find an approximate value for $\frac{\sqrt{0.9}}{1.1^2}$

b Find the absolute error of your estimate from part b.

c Find the percentage error of your estimate from part b.

13. Use Binomial Series to find values for the following, correct to 4 significant figures.

a $\sqrt{1.05}$

b $\sqrt{4.04}$

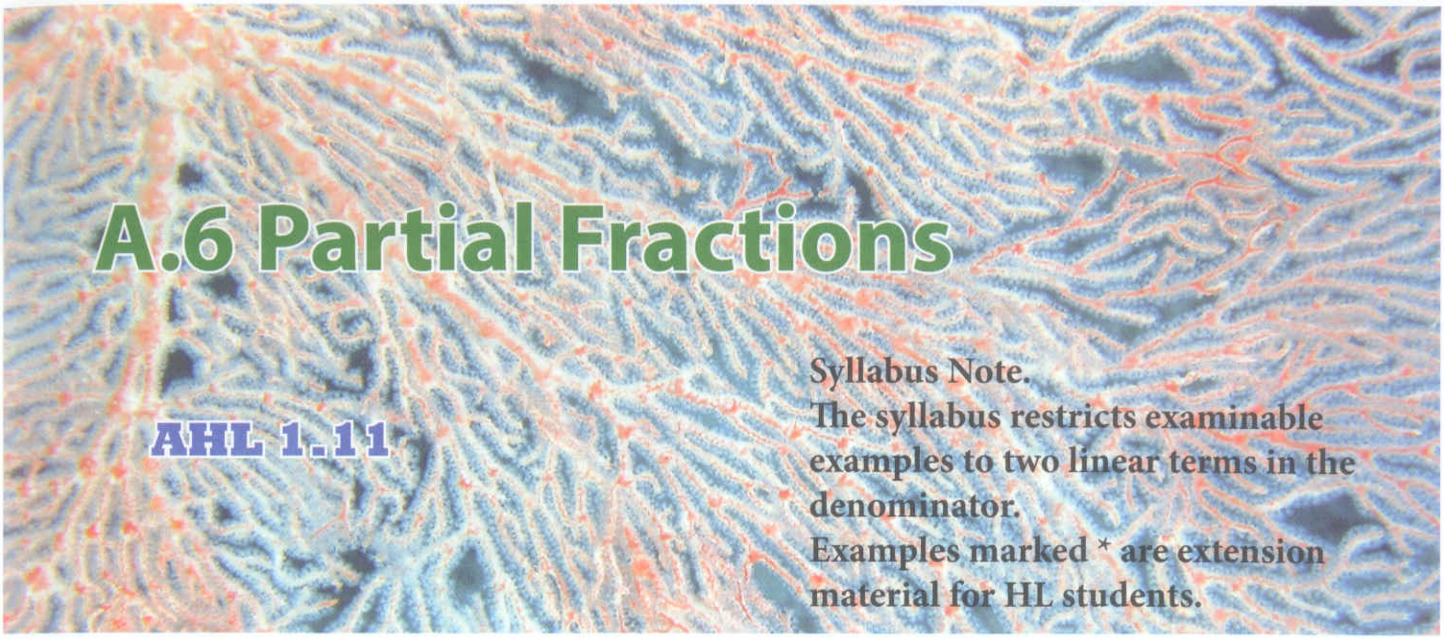
c $\frac{1}{1.01^3}$

d $1.01^9$

e $\frac{5}{\sqrt[6]{1.1}}$

f $\sqrt{202}$

Answers



# A.6 Partial Fractions

**AHL 1.11**

**Syllabus Note.**
**The syllabus restricts examinable examples to two linear terms in the denominator.**
**Examples marked * are extension material for HL students.**

Most of the calculations with fractions that you have learnt about in Middle School will likely have been additions and subtraction using common denominators.

$\frac{2}{5}+\frac{1}{4}=\frac{2\times 4}{5\times 4}+\frac{1\times 5}{4\times 5}=\frac{8}{20}+\frac{5}{20}=\frac{13}{20}$ or

$\frac{1}{6}+\frac{3}{4}=\frac{1\times 2}{6\times 2}+\frac{3\times 3}{4\times 3}=\frac{2}{12}+\frac{9}{12}=\frac{11}{12}$

In the first case, the common denominator is the product of the two denominators. Whilst this will always work, it is not always the **least** common denominator, as the second example shows.

The same technique can be applied to algebraic examples:

$$\begin{aligned}\frac{2}{x-1}+\frac{3}{x}&=\frac{2\times x}{(x-1)\times x}+\frac{3\times(x-1)}{x\times(x-1)}\\&=\frac{2x+3(x-1)}{x(x-1)}\\&=\frac{2x+3x-3}{x(x-1)}\\&=\frac{5x-3}{x(x-1)}\end{aligned}$$

This chapter will deal with reversing this process - splitting a single algebraic fractions into separate parts.

These separate parts are known as **partial fractions**.

It is not immediately obvious why one might want to do this. It is a technique that is used in calculus that enables us to increase the number of functions that we are able to integrate.

The technique depends on the type of fraction we are working on.

## Type 1: Degree of the numerator < degree of the denominator

**First case:** There is no repeated root in the denominator.

**Example A.6.1**

Express $\frac{1}{(x-2)(x+4)}, x\in\mathbb{R}\backslash\{2,-4\}$ as two partial fractions.

We will try to change the fraction to a sum of two fractions where the numerators are real numbers yet to be determined.

$$\frac{1}{(x-2)(x+4)}\equiv\frac{A}{x-2}+\frac{B}{x+4}, x\neq 2,-4$$

We want this to be an identity (true for all $x$ values except 2 & –4) - hence the identity sign. We will, however, revert to the more common equality sign from here on.

After rearranging the equation we get:

$$\begin{aligned}\frac{1}{(x-2)(x+4)}&\equiv\frac{A}{x-2}+\frac{B}{x+4}, x\neq 2,-4\\&=\frac{A(x+4)}{(x-2)(x+4)}+\frac{B(x-2)}{(x+4)(x-2)}\\&=\frac{A(x+4)+B(x-2)}{(x-2)(x+4)}\end{aligned}$$

$1=A(x+4)+B(x-2)$

We factorise $x$ out: $1=x(A+B)+4A-2B$ .

This could hold for any $x\in\mathbb{R}\backslash\{2,-4\}$. We need the solution of these equations $A+B=0$ and $4A-2B=1$.

The solution of this system of equations is $A=\frac{1}{6}, B=-\frac{1}{6}$.

Therefore:

$$\frac{1}{(x-2)(x+4)}=\frac{1}{6}\times\frac{1}{(x-2)}-\frac{1}{6}\times\frac{1}{(x+4)}$$
$$=\frac{1}{6(x-2)}-\frac{1}{6(x+4)}$$

We can check that our calculations are correct by adding the two fractions.

$$\frac{1}{6(x+4)}=\frac{1\times(x+4)}{6(x-2)\times(x+4)}-\frac{1\times(x-2)}{6(x+4)\times(x-2)}$$
$$=\frac{(x+4)-(x-2)}{6(x-2)(x+4)}$$
$$=\frac{6}{6(x-2)(x+4)}$$
$$=\frac{1}{(x-2)(x+4)}$$

**Example A.6.2**

Express $\frac{2x-5}{(x+2)(x-1)}$ as two partial fractions.

$$\frac{2x-5}{(x+2)(x-1)}=\frac{A}{(x+2)}+\frac{B}{(x-1)}$$
$$=\frac{A(x-1)}{(x+2)(x-1)}+\frac{B(x+2)}{(x-1)(x+2)}$$
$$2x-5=A(x-1)+B(x+2)$$
$$=(A+B)x-A+2B$$

$A+B=2$ and $-A+2B=-5$ has solution: $A=3$ and $B=-1$.

Therefore: $\frac{2x-5}{(x+2)(x-1)}=\frac{3}{(x+2)}-\frac{1}{(x-1)}$

**Example A.6.3**

Express $\frac{1}{x^2-x-6}$ as two partial fractions.

$$\frac{1}{x^2-x-6}=\frac{1}{(x-3)(x+2)}$$
$$=\frac{A}{x-3}+\frac{B}{x+2}$$
$$=\frac{A(x+2)+B(x-3)}{(x-3)(x+2)}$$

It follows that:

$$1=A(x+2)+B(x-3)\Rightarrow 1=Ax+2A+Bx-3B$$

This gives the two equations:

$$A+B=0, 2A-3B=1\Rightarrow A=\frac{1}{5}, B=-\frac{1}{5}$$

Therefore:

$$\frac{1}{x^2-x-6}=\frac{1}{5(x-3)}-\frac{1}{5(x+2)} \quad x\neq 3, -2.$$

**Second case:** There is repeated factor in the denominator:

**Example A.6.4**

Express $\frac{x-5}{(x+1)^2}$ as two partial fractions.

Examples of this type are handled by splitting the fraction into two as follows.

$$\frac{x-5}{(x+1)^2}=\frac{A}{x+1}+\frac{B}{(x+1)^2}$$

The best reason we can give for this is that it works!

$$\frac{x-5}{(x+1)^2}=\frac{A(x+1)}{(x+1)(x+1)}+\frac{B}{(x+1)^2}$$
$$=\frac{A(x+1)+B}{(x+1)^2}$$

It follows that: $x-5=A(x+1)+B$

$$=Ax+A+B$$

$1=A, A+B=-5 \Rightarrow B=-6$

We have the solution:

$$\frac{x-5}{(x+1)^2}=\frac{1}{x+1}-\frac{6}{(x+1)^2}$$

**Example A.6.5**

Express $\frac{3x^2+4x-6}{(x+2)^3}$ as partial fractions.

On this occasion we must use three partial fractions:

$$\frac{3x^2+4x-6}{(x+2)^3}=\frac{A}{(x+2)}+\frac{B}{(x+2)^2}+\frac{C}{(x+2)^3}$$

We proceed as before:

$$\frac{3x^2+4x-6}{(x+2)^3}=\frac{A}{(x+2)}+\frac{B}{(x+2)^2}+\frac{C}{(x+2)^3}$$

$$\frac{3x^2+4x-6}{(x+2)^3}=\frac{A(x+2)^2}{(x+2)^3}+\frac{B(x+2)}{(x+2)^3}+\frac{C}{(x+2)^3}$$

It follows that:

$$3x^2+4x-6=A\left(x^2+4x+4\right)+Bx+2B+C$$
$$=Ax^2+(4A+B)+4A+2B+C$$

This leads to three equations:

$$3=A$$
$$4=4A+B \Rightarrow B=-8$$
$$-6=4A+2B+C \Rightarrow C=-2$$

Therefore: $\frac{3x^2+4x-6}{(x+2)^3}=\frac{3}{(x+2)}-\frac{8}{(x+2)^2}-\frac{2}{(x+2)^3}$

**Third case:** repeated and non-repeated roots in the denominator.

**Example A.6.6**

Express $\frac{x-5}{(x+1)^2(x-1)}$ as partial fractions.

$$\frac{x-5}{(x+1)^2(x-1)}=\frac{A}{(x+1)}+\frac{B}{(x+1)^2}+\frac{C}{(x-1)}$$
$$=\frac{A(x+1)(x-1)+B(x-1)+C(x+1)^2}{(x+1)^2(x-1)}$$
$$=\frac{(A+C)x^2+(B+2C)x+(-A-B+C)}{(x+1)^2(x-1)}$$

This leads to three simultaneous equations:

$$A+C=0...[1]$$
$$B+2C=1...[2]$$
$$-A-B+C=-5...[3]$$

which are solved by linear combinations:

$$A+C=0...[1]$$
$$B+2C=1...[2]$$
$$-A-B+C=-5...[3]$$
$$[1]+[2]+[3]: 4C=-4 \Rightarrow C=-1$$
$$[1]: A=1$$
$$[2]: B-2=1 \Rightarrow B=3$$

Therefore: $\frac{x-5}{(x+1)^2(x-1)}=\frac{1}{(x+1)}+\frac{3}{(x+1)^2}-\frac{1}{(x-1)}$

**Example A.6.7**

Express $\frac{5x-3}{(x-3)^2(x-1)}$ as partial fractions.

We have the partial fractions split:

$$\frac{5x-3}{(x-3)^2(x-1)}=\frac{A}{(x-3)}+\frac{B}{(x-3)^2}+\frac{C}{(x-1)}$$

Proceeding as before:

$$\frac{5x-3}{(x-3)^2(x-1)}=\frac{A(x-3)(x-1)+B(x-1)+C(x-3)^2}{(x-3)^2(x-1)}$$
$$=\frac{(A+C)x^2+(-4A+B-6C)x+3A-B+9C}{(x-3)^2(x-1)}$$

The equations are:

$$A+C=0...[1]$$
$$-4A+B-6C=5...[2]$$
$$3A-B+9C=-3...[3]$$

$$[1] \Rightarrow A = -C$$

$$[2] \Rightarrow 4C + B - 6C = 5 \Rightarrow B - 2C = 5 \Rightarrow B = 2C + 5$$

$$[3] \Rightarrow -3C - (2C+5) + 9C = -3 \Rightarrow 4C - 5 = -3 \Rightarrow C = 0.5$$

$$B = 2C + 5 \Rightarrow B = 6$$

$$A = -C \Rightarrow A = -0.5$$

Therefore:

$$\frac{5x-3}{(x-3)^2(x-1)} = \frac{-0.5}{(x-3)} + \frac{6}{(x-3)^2} + \frac{0.5}{(x-1)}$$

Fourth case: irreducible (cannot be factorised) quadratic factor in the denominator.

In this case, we must have a linear numerator. Just a real number will not work.

**Example A.6.8***

Express $\frac{16}{x(x^2+4)}$ as partial fractions.

The split must be:

$$\frac{16}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$$

Note in particular the linear numerator paired with the irreducible quadratic denominator.

The solution proceeds:

$$\frac{16}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$$

$$= \frac{A(x^2+4) + (Bx+C)x}{x(x^2+4)}$$

$$= \frac{(A+B)x^2 + Cx + 4A}{x(x^2+4)}$$

$$A + B = 0, C = 0, 4A = 16$$

$$A = 4, B = -4, C = 0$$

$$\frac{16}{x(x^2+4)} = \frac{4}{x} - \frac{4x}{x^2+4}$$

You might like to see what happens if you try using a single number as the numerator of the second partial fraction. It will help you see why we adopt the above method.

**Example A.6.9***

Express $\frac{3x^2-4x}{(x-2)(x^2-x+2)}$ as partial fractions.

As in the previous example, the correct choice of numerators is important. Note that the degree of the numerator is less than that of the denominator.

$$\frac{3x^2-4x}{(x-2)(x^2-x+2)} = \frac{A}{x-2} + \frac{Bx+C}{x^2-x+2}$$

The solution proceeds as follows:

$$\frac{3x^2-4x}{(x-2)(x^2-x+2)} = \frac{A(x^2-x+2) + (Bx+C)(x-2)}{(x-2)(x^2-x+2)}$$

$$= \frac{(A+B)x^2 + (-A-2B+C)x + 2A - 2C}{(x-2)(x^2-x+2)}$$

This gives the equations:

$$A + B = 3 ...[1]$$

$$-A - 2B + C = -4 ...[2]$$

$$2A - 2C = 0 ...[3]$$

$$A = 1, B = 2, C = 1$$

so that: $\frac{3x^2-4x}{(x-2)(x^2-x+2)} = \frac{1}{x-2} + \frac{2x+1}{x^2-x+2}$

## Type 2 : if the degree of the numerator is equal to the degree of the denominator.

The method just discussed will not work in this case.

**Example A.6.10***

Express $\frac{x^2+1}{x^2-1}$ as partial fractions.

A necessary preliminary step is polynomial division. This works in a very similar way as division of numbers. If you are unfamiliar with this process, follow through this example first.

1
$$\begin{array}{rl} & x^2 \\ x-2 & \overline{)\,x^3-3x^2-10x+24} \end{array}$$
Divide $x^3$ by $x$ $(=x^2)$. Only look at the highest powers.

2
$$\begin{array}{rl} & x^2 \\ x-2 & \overline{)\,x^3-3x^2-10x+24} \\ & \;x^3-2x^2 \end{array}$$
Multiply the dividend $(x^2)$ by the divisor $(x-2)$ to get $x^3-2x^2$.

3
$$\begin{array}{rl} & x^2 \\ x-2 & \overline{)\,x^3-3x^2-10x+24} \\ & \;\underline{x^3-2x^2} \\ & \quad\; -x^2 \end{array}$$
Subtract $(x^3-3x^2-(x^3-2x^2)=-x^2)$ to get the remainder $-x^2$. Take care with signs!

4
$$\begin{array}{rl} & x^2 \\ x-2 & \overline{)\,x^3-3x^2-10x+24} \\ & \;\underline{x^3-2x^2}\downarrow \\ & \quad\; -x^2-10x \end{array}$$
Include the next column to the right - "bring down".

1
$$\begin{array}{rl} & x^2-x \\ x-2 & \overline{)\,x^3-3x^2-10x+24} \\ & \;\underline{x^3-2x^2}\downarrow \\ & \quad\; -x^2-10x \end{array}$$
REPEAT the 4 processes.

Divide $-x^2$ by $x$ $(=-x)$.

2
$$\begin{array}{rl} & x^2-x \\ x-2 & \overline{)\,x^3-3x^2-10x+24} \\ & \;\underline{x^3-2x^2}\downarrow \\ & \quad\; -x^2-10x \\ & \quad\; -x^2+2x \end{array}$$
Multiply the dividend $(-x)$ by the divisor $(x-2)$ to get $-x^2+2x$.

3
$$\begin{array}{rl} & x^2-x \\ x-2 & \overline{)\,x^3-3x^2-10x+24} \\ & \;\underline{x^3-2x^2}\downarrow \\ & \quad\; -x^2-10x \\ & \quad\; \underline{-x^2+2x} \\ & \qquad\quad -12x \end{array}$$
Subtract $(-x^2-10x-(-x^2+2x)=-12x)$ to get the remainder $-12x$.

4
$$\begin{array}{rl} & x^2-x \\ x-2 & \overline{)\,x^3-3x^2-10x+24} \\ & \;\underline{x^3-2x^2}\downarrow \\ & \quad\; -x^2-10x \\ & \quad\; \underline{-x^2+2x} \\ & \qquad\quad -12x+24 \end{array}$$
Include the next column to the right - "bring down".

1
$$\begin{array}{rl} & x^2-x-12 \\ x-2 & \overline{)\,x^3-3x^2-10x+24} \\ & \;\underline{x^3-2x^2}\downarrow \\ & \quad\; -x^2-10x \\ & \quad\; \underline{-x^2+2x} \\ & \qquad\quad -12x+24 \end{array}$$
REPEAT these processes.

Divide $-12x$ by $x$ $(=-12)$.

2
$$\begin{array}{rl} & x^2-x-12 \\ x-2 & \overline{)\,x^3-3x^2-10x+24} \\ & \;\underline{x^3-2x^2}\downarrow \\ & \quad\; -x^2-10x \\ & \quad\; \underline{-x^2+2x} \\ & \qquad\quad -12x+24 \\ & \qquad\quad -12x+24 \end{array}$$
Multiply the dividend $(-12)$ by the divisor $(x-2)$ to get $-12x+24$.

3
$$\begin{array}{rl} & x^2-x-12 \\ x-2 & \overline{)\,x^3-3x^2-10x+24} \\ & \;\underline{x^3-2x^2}\downarrow \\ & \quad\; -x^2-10x \\ & \quad\; \underline{-x^2+2x} \\ & \qquad\quad -12x+24 \\ & \qquad\quad \underline{-12x+24} \\ & \qquad\qquad\quad 0 \end{array}$$
Subtract $(-12x+24-(-12x+24)=0)$ to get the remainder 0.

In our present case, the division looks like this:

$$\begin{array}{rl} & 1 \\ x^2-1 & \overline{)\,x^2+1} \\ & \;\underline{x^2-1} \\ & \qquad 2 \end{array}$$

From this it follows that: $\dfrac{x^2+1}{x^2-1}=1+\dfrac{2}{x^2-1}$

The same result can be arrived at as follows:

$$\frac{x^2+1}{x^2-1}=1+\frac{2}{x^2-1}$$

$$\begin{aligned}\frac{x^2+1}{x^2-1}&=\frac{x^2-1+2}{x^2-1}\\ &=\frac{x^2-1}{x^2-1}+\frac{2}{x^2-1}\\ &=1+\frac{2}{x^2-1}\end{aligned}$$

The second part of the expression can now be split into partial fractions using techniques already discussed.

$$\frac{x^2+1}{x^2-1}=1+\frac{1}{x-1}-\frac{1}{x+1}$$

## Type 3: if the degree of the numerator is bigger than the degree of the denominator

**Example A.6.11***

Express $\dfrac{x^3-3x^2+1}{x^2-x-2}$ as partial fractions.

As with the previous type, it is necessary to split the fraction into polynomial parts and a fractional part that has a numerator of degree less than the denominator.

$$\begin{array}{r|l} & x \quad -2 \\ x^2-x-2 & x^3-3x^2 \qquad +1 \\ & x^3-\ x^2-2x \\ \hline & -2x^2+2x+1 \\ & -2x^2+2x+4 \\ \hline & -3 \end{array}$$

$$\frac{x^3-3x^2+1}{x^2-x-2}=x-2+\frac{-3}{x^2-x-2}$$

This result can also be arrived at:

$$\begin{aligned}\frac{x^3-3x^2+1}{x^2-x-2}&=\frac{(x^2-x-2)(x-2)-3}{x^2-x-2}\\&=\frac{(x^2-x-2)(x-2)}{x^2-x-2}+\frac{-3}{x^2-x-2}\\&=x-2+\frac{-3}{x^2-x-2}\end{aligned}$$

We must now split the remainder term into partial fractions:

$$\begin{aligned}\frac{-3}{x^2-x-2}&=\frac{-3}{(x-2)(x+1)}\\&=\frac{A}{x-2}+\frac{B}{x+1}\end{aligned}$$

This can now be solved as before:

$$\frac{-3}{(x-2)(x+1)}=\frac{A(x+1)+B(x-2)}{(x-2)(x+1)}$$

$$-3=(A+B)x+A-2B$$

$$A+B=0...[1]$$

$$A-2B=-3...[2]$$

$$[1]-[2]{:}3B=3\Rightarrow B=1$$

$$[1]{:}A+1=0\Rightarrow A=-1$$

$$\frac{-3}{(x-2)(x+1)}=\frac{-1}{x-2}+\frac{1}{x+1}$$

The complete result is:

$$\frac{x^3-3x^2+1}{x^2-x-2}=x-2+\frac{-1}{x-2}+\frac{1}{x+1}$$

We conclude with some miscellaneous examples.

**Example A.6.12**

Express $\frac{1}{x(x+1)}$ as partial fractions.

This is a 'type one' example.

$$\begin{aligned}\frac{1}{x(x+1)}&=\frac{A}{x}+\frac{B}{x+1}\\&=\frac{A(x+1)+Bx}{x(x+1)}\\&=\frac{(A+B)x+A}{x(x+1)}\end{aligned}$$

$$A+B=0...[1]$$

$$A=1...[2]$$

$$B=-1$$

$$\frac{1}{x(x+1)}=\frac{1}{x}-\frac{1}{x+1}$$

**Example A.6.13***

Express $\frac{2x^2+5x-2}{x+3}$ as partial fractions.

This example requires division.

$$\begin{array}{r|l} & 2x \ -1 \\ x+3 & 2x^2+5x-2 \\ & 2x^2+6x \\ \hline & \quad -x-2 \\ & \quad -x-3 \\ \hline & \qquad 1 \end{array}$$

This means that:

$$\frac{2x^2+5x-2}{x+3}=2x-1+\frac{1}{x+3}$$

which completes the question.

**Example A.6.14***

Express $\frac{x+1}{(x^2+1)(x-1)}$ as partial fractions.

This example requires the correct split:

$$\frac{x+1}{(x^2+1)(x-1)}=\frac{A}{x-1}+\frac{Bx+C}{x^2+1}$$

The solution is:

$$\frac{x+1}{(x^2+1)(x-1)}=\frac{A}{x-1}+\frac{Bx+C}{x^2+1}$$

$$\frac{x+1}{(x^2+1)(x-1)}=\frac{A(x^2+1)+(Bx+C)(x-1)}{(x^2+1)(x-1)}$$

$$=\frac{(A+B)x^2+(-B+C)x+A-C}{(x^2+1)(x-1)}$$

$$A+B=0...[1]$$

$$-B+C=1...[2]$$

$$A-C=1...[3]$$

$$[1]+[2]+[3]:2A=2\Rightarrow A=1$$

$$[1]:1+B=0\Rightarrow B=-1$$

$$[3]:1-C=1\Rightarrow C=0$$

$$\frac{x+1}{(x^2+1)(x-1)}=\frac{1}{x-1}+\frac{-x}{x^2+1}$$

## Exercise A.6.1

1. Express as partial fractions:

a $\frac{5x}{(x+2)(x-3)}$

b $\frac{3x+1}{(x+4)(x-7)}$

c $\frac{-x-13}{(x+1)(x-3)}$

d $\frac{-3x-5}{(2x+1)(x-3)}$

e $\frac{7x+5}{(2x+3)(3x-1)}$

2. Express as partial fractions:

a $\frac{3x-1}{(x-1)^2}$

b $\frac{-2x+10}{(x-3)^2}$

c $\frac{-2x^2+8x-5}{(x-1)^3}$

d $\frac{(3-x)(2x-3)}{(x-2)^3}$

e $\frac{16x^2+18x+8}{(2x+1)^3}$

3. Express as partial fractions:

a $\frac{2x^2+3x+4}{(x+2)(x^2+2)}$

b $\frac{x^2+3x+11}{(x+2)(x^2+x+7)}$

c $\frac{3x^2+5x+1}{(2x+1)(x^2+3x+1)}$

d $\frac{x^2+3x+13}{(x+2)(x^2+7)}$

e $\frac{7x^2+13x+10}{(x+5)(x^2+5)}$

4. Express as partial fractions:

a $\frac{3x+1}{x+1}$

b $\frac{3x^2+x+6}{x^2+2}$

c $\frac{2x^3-x^2+11x-5}{x^2+5}$

d $\dfrac{x^3-4x^2+5x+1}{(x-1)^2}$

e $\dfrac{2x^3-x^2-2x+3}{2x-1}$

5. Express as partial fractions:

a $\dfrac{x+1}{(x^2+1)(x-1)}$

b $\dfrac{x}{(x^2+3)(x^2+5)}$

c $\dfrac{x^3-3x-4}{x^3-1}$

d $\dfrac{3x+1}{(x-1)(x^2+x+1)}$

e $\dfrac{x^3}{x^2-4}$

Answers

# A.7 Complex Numbers

**AHL 1.12-14**

## Introduction

Complex numbers are often first encountered when solving a quadratic equation of the type for which there are no real solutions, e.g. $x^2+1=0$ or $x^2+2x+5=0$ (because for both equations the discriminant, $\Delta=b^2-4ac$, is negative). However, the beginning[1] of complex numbers is to be found in the work of *Girolamo Cardano* (1501–1576), who was resolving a problem which involved the solution to a reduced cubic equation of the form $x^3+ax=b, a>0, b>0$. Although others later improved on the notation and the mechanics of complex algebra, it was the work found in his book, *Ars magna*, that led to the common usage of complex numbers found today.


## Notation and $i^2 = -1$

The set of complex numbers is denoted by:

$$C=\{z : z=x+iy, \text{ where } x,y\in\mathbb{R}, i^2=-1\}$$

The complex number, $z$, is 'made up' of two parts; '$x$' and '$iy$'. The '$x$-term' is called the **real part** and the '$y$-term' is the **imaginary part** i.e. the part attached to the '$i$', where $i^2=-1$. It is important to note the following:

1. The complex number $z=x+iy$ is a single number (even though there are 'two parts', it is still a single value).

2. **The real part** of $z$, denoted by $Re(z)$ is $x$.

   **The imaginary** part of $z$, denoted by $Im(z)$ is $y$.

This means that the complex number $z$ can be written as:

$$z=Re(z)+Im(z)i$$

Notice that the imaginary part is not '$iy$' but simply '$y$'.

### Example A.7.1

For each of the following complex numbers, state the real and imaginary parts of:

a $z=2+3i$ b $w=3-9i$

a We have that $Re(z)=Re(2+3i)=2$ and $Im(z)=Im(2+3i)=3$.

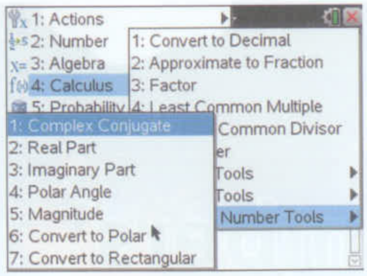
Therefore, the real part of $z$ is 2 and the imaginary part of $z$ is 3.

b Similarly, $Re(w)=Re(3-9i)=3$ and $Im(w)=Im(3-9i)=-9$.

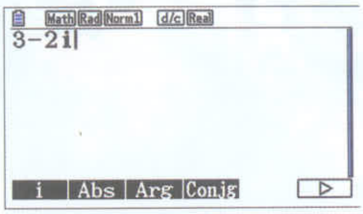
That is, for $w$, the real part is 3 and the imaginary part is –9.

1 See *An Imaginary Tale, The Story of*, by Paul J. Nahim.

It is important to locate, and become familiar with, the Complex Number part of your calculator (TI).




If using Casio, press the Option key (OPTN) followed by F3 - complex,




**Example A.7.2**

If $z = 2xi + y^2 - 1$, find the values of $x$ and $y$ for which $Im(z) = 8$ and $Re(z) = 0$.

We first need to determine what the real and imaginary parts of $z = 2xi + y^2 - 1$ are.

We have that $Im(z) = Im[(2x)i + (y^2 - 1)] = 2x$.

$\therefore Im(z) = 8 \Leftrightarrow 2x = 8 \Leftrightarrow x = 4$.

Similarly, $Re(z) = Re[(2x)i + (y^2 - 1)] = y^2 - 1$

$\therefore Re(z) = 0 \Leftrightarrow y^2 - 1 = 0 \Leftrightarrow y = \pm 1$.

# The algebra of complex numbers

## Working with 'i'

Since we have that $i = \sqrt{-1}$, then $i^2 = -1$, meaning that $i^3 = i^2 \times i = -1 \times i = -i$.

Similarly, $i^4 = i^2 \times i^2 = -1 \times -1 = 1$, etc.

General results for expressions such as $i^n$ can be determined. We leave this to the set of exercises at the end of this section.

## Operations

For any two complex numbers $z_1 = a + ib$ and $z_2 = c + id$, the following hold true:

### Equality:

Two complex numbers are equal if and only if their real parts are equal **and** their imaginary parts are equal.

$$z_1 + z_2 = (a+ib) + (c+id) = (a+c) + (b+d)i$$

### Addition:

The sum of two (or more) complex numbers is made up of the sum of their real parts plus the sum of their imaginary parts (multiplied by '$i$').

$$z_1 + z_2 = (a+ib) = (c+id) = (a+c) + (b+d)i$$

### Subtraction:

The difference of two (or more) complex numbers is made up of the difference of their real parts plus the difference of their imaginary parts (multiplied by '$i$').

$$z_1 - z_2 = (a+ib) - (c+id) = (a-c) + (b-d)i$$

### Multiplication:

When multiplying two (or more) complex numbers, we complete the operation as we would with normal algebra. However, we use the fact that $i^2 = -1$ when simplifying the result.

$$\begin{aligned} z_1 z_2 &= (a+ib)(c+id) \\ &= ac + adi + bci + bdi^2 \\ &= (ac - bd) + (ad + bc)i \end{aligned}$$

### Conjugate:

The conjugate of $z = x + iy$, denoted by $\bar{z}$ or $z^*$ is the complex number $z^* = x - iy$. Note that:

$$\begin{aligned} zz^* &= (x+iy)(x-iy) \\ &= x^2 - xyi + xyi - y^2 i^2 \\ &= x^2 + y^2 \end{aligned}$$

That is, when a complex number is multiplied with its conjugate, the result is a real number. $z = x + iy$ and $z^* = x - iy$ are known as **conjugate pairs**.

## Division:

When dividing two complex numbers, we multiply the numerator and denominator by the conjugate of the denominator (this has the effect of 'realizing' the denominator).

That is,

$$\begin{aligned}\frac{z_1}{z_2} &= \frac{a+ib}{c+id}\\ &= \frac{a+ib}{c+id}\times\frac{c-id}{c-id}\\ &= \left(\frac{ac+bd}{c^2+d^2}\right)+\left(\frac{bc-ad}{c^2+d^2}\right)i\end{aligned}$$

Note: It is important to realise that these results are not meant to be memorised. Rather, you should work through the multiplication or division in question and then simplify the result.

### Example A.7.3

Find the values of $x$ and $y$ if
$z = x+(y-2)i,\ w = 4+i$ and $z = w$.

Recall: Two complex numbers are equal if and only if their corresponding real parts and imaginary parts are equal.

So, $z = w \Leftrightarrow x+(y-2)i = 4+i \Leftrightarrow x = 4$ and $y-2 = 1$.

That is, $z = w$ if and only if $x = 4$ and $y = 3$.

### Example A.7.4

Find the real values $x$ and $y$ given that
$$(3-2i)(x+iy) = 12-5i.$$

As we are equating two complex numbers, we need to determine the simultaneous solution brought about by equating their real parts and imaginary parts:

From $(3-2i)(x+iy) = 12-5i$ we have
$3x+3yi-2xi-2yi^2 = 12-5i$
$\Leftrightarrow (3x+2y)+(3y-2x)i = 12-5i$

$\Leftrightarrow 3x+2y = 12$ – (1) and $3y-2x = -5$ – (2)

Solving simultaneously, we have:

$2\times$(1): $6x+4y = 24$ – (3)

$3\times$(2): $9y-6x = -15$ – (4)

Adding, (3) + (4), we have: $13y = 9$

Therefore, $y = \frac{9}{13}$. Then, substituting into (1) we have:

$3x+2\times\frac{9}{13} = 12 \Leftrightarrow 3x = \frac{138}{13} \Leftrightarrow x = \frac{46}{13}$.

So, we have the solution pair, $x = \frac{46}{13}, y = \frac{9}{13}$.

### Example A.7.5

Given that $z = 3+i$ and $w = 1-2i$, evaluate the following.

a $z+w$   b $2z-3w$

c $zw$   d $w^2$

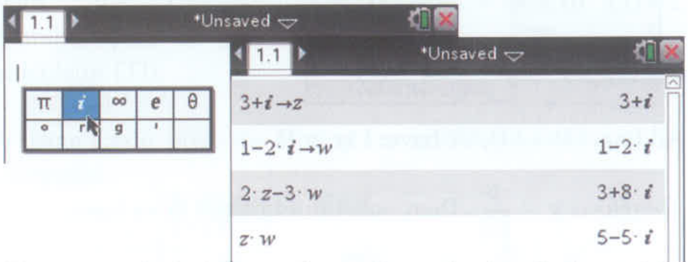
a $$\begin{aligned}z+w &= (3+i)+(1-2i)\\ &= (3+1)+(i-2i)\\ &= 4-i\end{aligned}$$

b $$\begin{aligned}2z-3w &= 2(3+i)-3(1-2i)\\ &= (6-3)+(2i+6i)\\ &= 3+8i\end{aligned}$$

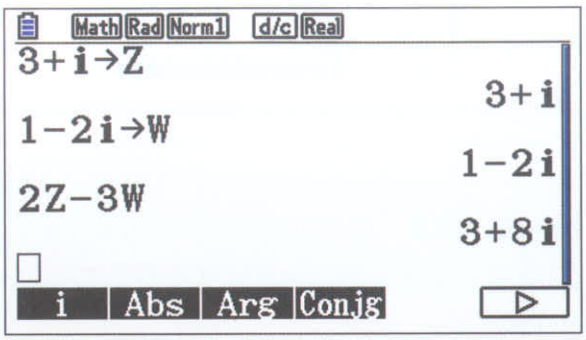
c $$\begin{aligned}zw &= (3+i)(1-2i)\\ &= 3-6i+i-2i^2\\ &= 3-5i+2\\ &= 5-5i\end{aligned}$$

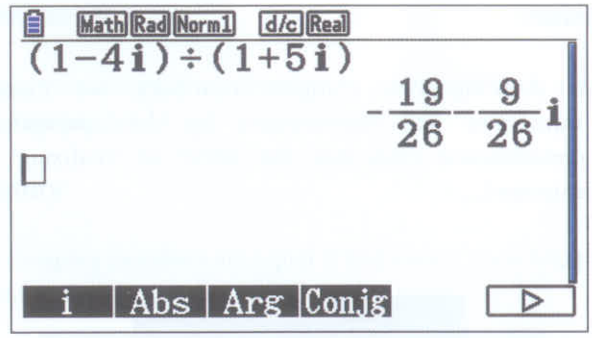
d $$\begin{aligned}w^2 &= (1-2i)(1-2i)\\ &= 1-2i-2i+4i^2\\ &= 1-4i-4\\ &= -3-4i\end{aligned}$$

You should be able to perform such calculations both manually and using your calculator. Parts b & c of the previous example are solved as follows (note that you must use the complex number version of '$i$', not the variable 'I').




The same calculations can be performed using Casio models:




You may need to use the F↔D key (above 8) to get the answer as a fraction.

**Example A.7.6**

Find the conjugate of:

a $z = 2+6i$ b $w = \sqrt{3}i - 1$

a $z = 2+6i \Rightarrow z^* = \overline{2+6i} = 2-6i$.

b $w = \sqrt{3}i - 1 \Rightarrow w^* = \overline{\sqrt{3}i-1} = -\sqrt{3}i - 1$.

**Example A.7.7**

Express the complex number $\frac{1-4i}{1+5i}$ in the form $u+iv$.

$$\begin{aligned}\frac{1-4i}{1+5i} &= \frac{1-4i}{1+5i}\times\frac{1-5i}{1-5i}\\ &= \frac{1-5i-4i+20i^2}{1-5i+5i-25i^2}\\ &= \frac{1-9i-20}{1+25}\\ &= \frac{-19-9i}{26}\\ &= -\frac{19}{26}-\frac{9}{26}i\end{aligned}$$

**Example A.7.8**

If $z = x+iy$, find:

a $z+z^*$ b $z-z^*$

$z+z^* = (x+iy)+(x-iy) = 2x = 2Re(z)$.

Note then, that $Re(z) = \frac{1}{2}(z+z^*)$.

$z-z^* = (x+iy)-(x-iy) = 2yi = 2Im(z)i$

Note then, that $Im(z) = \frac{1}{2i}(z-z^*)$.

**Example A.7.9**

If $z = \cos\theta + i\sin\theta$ and $w = \sin\alpha + i\cos\alpha$ express $zw$ in the form $p+qi$, where $p, q \in \mathbb{R}$. Hence find the maximum value of $p^2+q^2$.

$$\begin{aligned}zw &= (\cos\theta + i\sin\theta)(\sin\alpha + i\cos\alpha)\\ &= \cos\theta\sin\alpha + \cos\theta\cos\alpha i + \sin\theta\sin\alpha i + \sin\theta\cos\alpha i^2\\ &= \cos\theta\sin\alpha + \cos\theta\cos\alpha i + \sin\theta\sin\alpha i - \sin\theta\cos\alpha\\ &= (\cos\theta\sin\alpha - \sin\theta\cos\alpha) + (\cos\theta\cos\alpha + \sin\theta\sin\alpha)i\\ &= \sin(\alpha-\theta) + \cos(\alpha-\theta)i\end{aligned}$$

With $p = \sin(\alpha-\theta)$ and $q = \cos(\alpha-\theta)$

we have $p^2 + q^2 = \sin^2(\alpha - \theta) + \cos^2(\alpha - \theta) = 1$.

As $p^2 + q^2$ will always have a fixed value of 1, its maximum value is also 1.

## Exercise A.7.1

1. Find: a $Re(z)$ b $Im(z)$ c $z^*$ for each of the following.

   i $z = 2 + 2i$ ii $z = -3 + \sqrt{2}i$

   iii $z = -5i + 6$ iv $z = -\frac{2}{5}i$

   v $z = \frac{3+i}{2}$ vi $2z = 1 - 3i - z$

2. If $z = 4 - i$ and $w = 3 + 2i$, find in simplest form (i.e. expressed as $u + iv$), the following.

   a $z + w$ b $z - w$ c $z^2$

   d $2z - 3w$ e $z^*w$ f $iw$

3. If $z = 2 + i$ and $w = -3 + 2i$, find in simplest form (i.e. expressed as $u + iv$), the following.

   a $z + w$ b $z - w$ c $iz^2$

   d $z^2 - i^2w$ e $\bar{z}w$ f $\overline{iw}$

4. For the complex numbers $z = 1 - i$ and $w = 2i - 3$, express each of the following in the form $u + iv$.

   a $\frac{1}{z}$ b $\frac{w}{z}$ c $\frac{z+1}{i}$

   d $z^{-2}$ e $\frac{2i}{w+3}$ f $\frac{z^*}{w^*}$

5. Simplify the following.

   a $(2 + 4i)(3 - 2i)$ b $(1 - i)^3$

   c $(1 + \sqrt{2}i)^2 i$ d $\frac{i}{1 + 2i}$

   e $\frac{1 + 2i}{i}$ f $\frac{(1 - i)i}{(-i + 2)}$

6. Given that $z = 3 + \sqrt{2}i$ and $w = \frac{1}{1-i}$, find:

   a $Re(w)$ b $Im(zw)$ c $Re\left(\frac{z}{w}\right)$

7. Find the real numbers $x$ and $y$ such that:

   a $2x + 3i = 8 - 6yi$

   b $x + iy = (2 + 3i)^2$

   c $(x + iy)(-i) = 5$

8. a Simplify $i^n$ for:

   i $n = 0, 1, 2, 3, 4, 5$

   ii $n = -1, -2, -3, -4, -5$

   Evaluate :

   i $i^{10}$ ii $i^{15}$

   iii $i^{90}$ iv $i^{74}$

9. Find the real numbers $x$ and $y$, for which $(x + yi)(5 - 2i) = -18 + 15i$.

10. Show that for any complex numbers $z = x + iy$ and $w = a + bi$:

    a $(z + w)^* = z^* + w^*$ b $(z - w)^* = z^* - w^*$

    c $(zw)^* = z^*w^*$ d $(z^2)^* = (z^*)^2$

    e $\left(\frac{z}{w}\right)^* = \frac{z^*}{w^*}$ f $(z^*)^* = z$

11. a Prove that $z\bar{w} - \bar{z}w$ is purely imaginary or zero for all complex numbers $z$ and $w$.

    b Prove that $z\bar{w} + \bar{z}w$ is real for all complex numbers $z$ and $w$.

12. Given that $w = \frac{z-1}{\bar{z}+1}$, where $z = x + iy$,

find the condition(s) under which:

a $w$ is real b $w$ is purely imaginary.

13. a Find the real values of $x$ and $y$, such that $(x+iy)^2 = 8-6i$.

b Hence, determine $\sqrt{8-6i}$, expressing your answer in the form $u+iv$, where $u$ and $v$ are both real numbers and $u > 0$. Find $\sqrt{3-4i}$, expressing your answer in the form $u+iv$, where $u$ and $v$ are both real numbers and $u > 0$.

14. Simplify the following.

a $(1+i)^3 - (1-i)^3$

b $(1+i)^3 + (1-i)^3$

c $\frac{(1+i)^3}{(1-i)^3}$

15. Find the real values $x$ and $y$ for which:

**a** $(x-y) + 4i = 9 + yi$

**b** $(2x+3y) - x^3 i = 12 - 64i$.

16. Find the complex number $z$ given that:

$5z + 2i = 5 + 2iz$,

giving your answer in the form $a + ib$, where $a$ and $b$ are real.

17. Find the complex number $z$ which satisfies the equation $z(1+\sqrt{2}i) = 1-\sqrt{2}i$.

18. The complex number $z$ satisfies the equation $z^2 - i = 2z - 1$. If $z = u + iv$ find all real values of $u$ and $v$.

19. If $z = \frac{2-i}{1+i}$, find:

a $Re(z^2) + Im(z^2)$

b $Re\left(z+\frac{1}{z}\right) + Im\left(z+\frac{1}{z}\right)$

20. a Show that:

i $\frac{1+i}{1-i} = i$

ii $\left(\frac{1+i}{1-i}\right)^2 = -1$

b Show that $\left(\frac{1+i}{1-i}\right)^{4k} = 1$ if $k$ is a positive integer.

21. Find the complex number(s) $z = a + bi$,

satisfying the equation $\frac{1+z^2}{1-z^2} = i$.

22. Express the following in the form $p + qi$, where $p$ and $q$ are real numbers.

a $(\cos\theta + i\sin\theta)(\cos\alpha + i\sin\alpha)$

b $(\cos\theta + i\sin\theta)(\cos\alpha - i\sin\alpha)$

c $(r_1\cos\theta + ir_1\sin\theta)(r_2\cos\alpha + ir_2\sin\alpha)$

d $(x - \cos\theta - i\sin\theta)(x - \cos\theta + i\sin\theta)$

e $(x + \sin\alpha + i\cos\alpha)(x + \sin\alpha - i\cos\alpha)$

23. For the complex number defined as $z = \cos(\theta) + i\sin(\theta)$, show that:

a $z^2 = \cos(2\theta) + i\sin(2\theta)$

b $z^3 = \cos(3\theta) + i\sin(3\theta)$

Assuming now that $z^k = \cos(k\theta) + i\sin(k\theta)$, show that:

c $C + i(S-1) = \dfrac{1-z^n}{1-z}$,

where $C = 1 + \cos(\theta) + \cos(2\theta) + \ldots + \cos((n-1)\theta)$

and

$S = 1 + \sin(\theta) + \sin(2\theta) + \ldots + \sin((n-1)\theta)$,

where $0 < \theta < \dfrac{\pi}{2}$.

24. a Given that $(x+iy)^2 = 8+6i$, find the values of $x$ and $y$. Hence, find $\sqrt{8+6i}$.

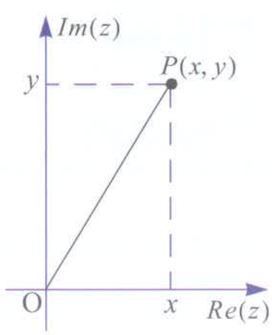
b If $(2+3i)(3-4i) = p+qi$, find the value of $p^2+q^2$.

c If $(x+iy)^2 = a+ib$, find an expression for $a^2+b^2$ in terms of $x$ and $y$.

Extra questions

## The Argand Diagram

Unlike real numbers (which can be described geometrically by the position they occupy on a one dimensional number line), complex numbers require the real and imaginary parts to be described. The geometrical representation best suited for this purpose would be two dimensional. Any complex number $z = x+iy$ may be represented on an **Argand Diagram**, by using either




1. the point $P(x, y)$, or
2. the position vector $\overrightarrow{OP}$

That is, we make use of a plane that is similar to the standard Cartesian plane to represent the complex number $z = x+iy$. This means that the $x$-axis represents the $Re(z)$ value and the $y$-axis represents the $Im(z)$ value.

*The complex plane has led to the Mandelbrot Set (heading picture by Binette228) and models of tree branching and other elaborate natural forms.*

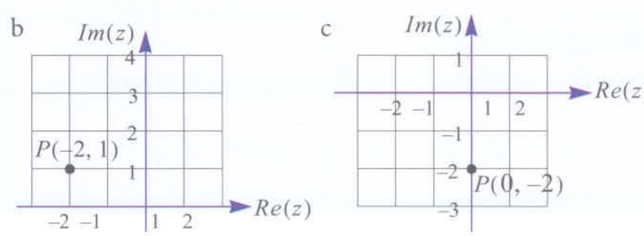
**Example A.7.10**

Represent each of the following complex numbers on an Argand diagram.

a $z = 1+3i$ b $z = -2+i$ c $z = -2i$.

a With $z = 1+3i$, we have $x = Re(z) = 1$ and $y = Im(z) = 3$. Therefore, we may represent the complex number $z = 1+3i$ by the point $P(1,3)$ on the Argand diagram:

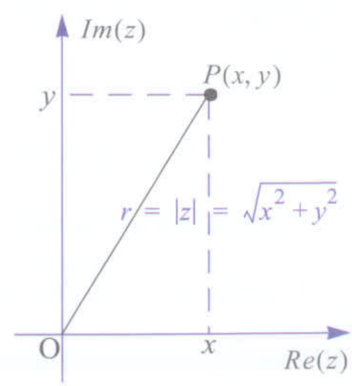
Similarly for parts b and c we have:




### Geometrical properties of complex numbers

#### The modulus of z

The modulus of a complex number $z = x+iy$ is a measure of the length of $z = x+iy$ and is denoted by $|z|$. That is, $mod(z) = |z|$.




The **modulus** of $z$ is also called the **magnitude** of $z$. We can determine the length by using Pythagoras's theorem:

$$(OP)^2 = x^2 + y^2$$

$$\therefore OP = \sqrt{x^2 + y^2}$$

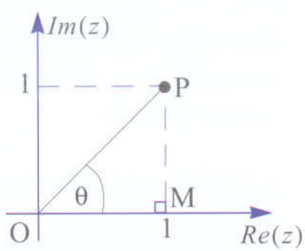
That is, if $x = x + iy$ then $mod(z) = |z| = \sqrt{x^2 + y^2}$.

The modulus of $z$ is also written as $r$, i.e. $r = |z|$.

Notice then, that $|z| = \sqrt{x^2 + y^2} = \sqrt{zz^*}$

## The Argument of z

The **argument** of a complex number $z = x + iy$ is a measure of the angle which $z = x + iy$ makes with the **positive** $Re(z)$-axis and is denoted by $\arg(z)$ and sometimes by $ph(z)$, which stands for the **phaze** of $z$. If $\theta$ is this angle, we then write, $\theta = \arg(z)$.




If $-\pi < \theta < \pi$, then $\theta = Arg(z)$

Notice the use of capital '$A$' rather than lower case '$a$'. Using $\theta = Arg(z)$, implies that we are referring to the **Principal argument value**, that is, we have restricted the range in which the angle $\theta$ lies.

**Example A.7.11**

Find the modulus of the following complex numbers.

a $z = 4 + 3i$ b $z = -1 + 2i$.

a $z = 4 + 3i \therefore |z| = \sqrt{(4)^2 + (3)^2} = \sqrt{25} = 5$

Notice that we only square the real and imaginary parts of the complex number. That is, we do not use $3i$ because this would give $\sqrt{(4)^2 + (3i)^2} = \sqrt{16 - 9} = \sqrt{7}$!

b In the same way we have:

$z = -1 + 2i \therefore |z| = \sqrt{(-1)^2 + (2)^2} = \sqrt{5}$.

**Example A.7.12**

Find the principal argument of the following complex numbers.

a $z = 1 + i$ b $z = -1 + 2i$. c $-1 - \sqrt{3}i$

When finding the principal argument of a complex number, an Argand diagram can be used as an aid. This will always enable us to work with right–angled triangles. Then we can make use of the diagram to find the restrictions on the required angle, i.e. $-\pi < \theta \le \pi$, then $\theta = Arg(z)$.

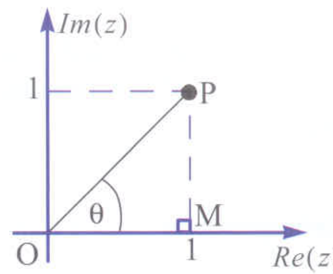
a We first represent $z = 1 + i$ on an Argand diagram:

From the triangle OPM, we have:

$$\tan\theta = \frac{PM}{OM} = \frac{1}{1}$$

$$\therefore \theta = Tan^{-1}(1)$$

$$\Rightarrow \theta = \frac{\pi}{4} \text{ (or } 45°\text{)}$$




Therefore, the principal argument of $z$, is $Arg(z) = \frac{\pi}{4}$.

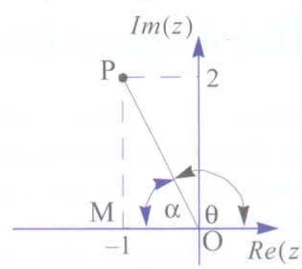
b Again, we start by using an Argand diagram:

From the triangle OPM, we have:

$$\tan\alpha = \frac{PM}{OM} = \frac{2}{1}$$

$$\therefore \alpha = Tan^{-1}(2)$$

$$\Rightarrow \alpha = 63°26'$$




Therefore, $\theta = 180 - 63°26' = 116°34'$.

So that (the principal argument) $Arg(z) = 116°34'$.

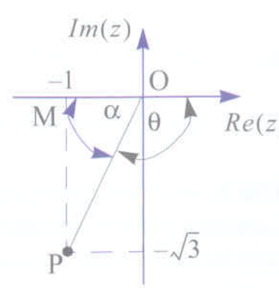
c Notice that we only make use of $\alpha$ to help us determine $\theta$ [i.e. $\alpha + \theta = \pi$ ( or $180°$)]

From the triangle OPM, we have:

$$\tan\alpha = \frac{PM}{OM} = \frac{\sqrt{3}}{1}$$

$$\therefore \alpha = Tan^{-1}(\sqrt{3})$$

$$\Rightarrow \alpha = 60°$$




Therefore, $\theta = 180 - 60° = 120°$.

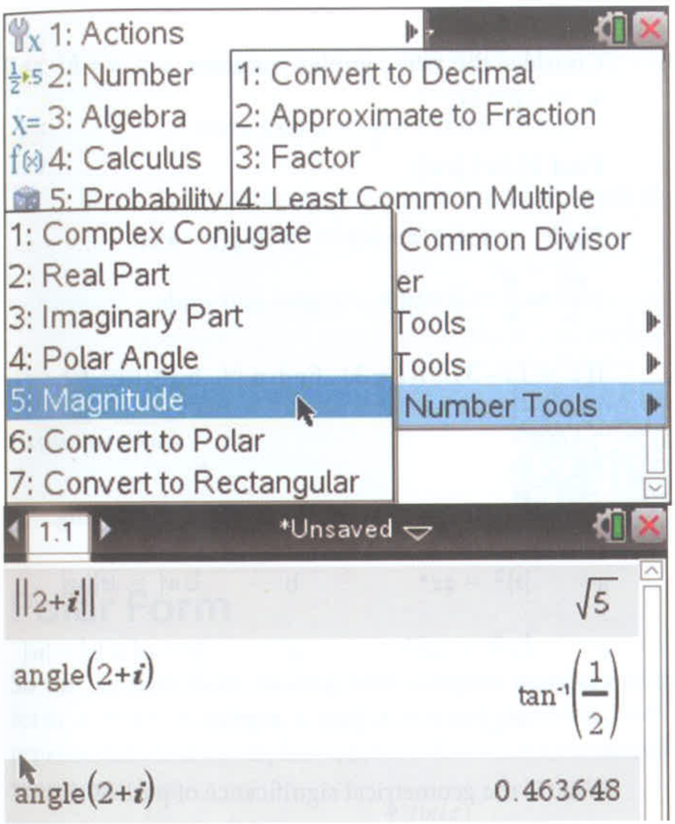
So that (the principal argument) $Arg(z) = -120°$.

Notice that because we are 'moving' in a clockwise direction, the angle is negative.

Notice that in the last example, although $Arg(z) = -120°$, we could have written $\arg(z) = 180° + 60° = 240°$ (using 'small' '$a$').

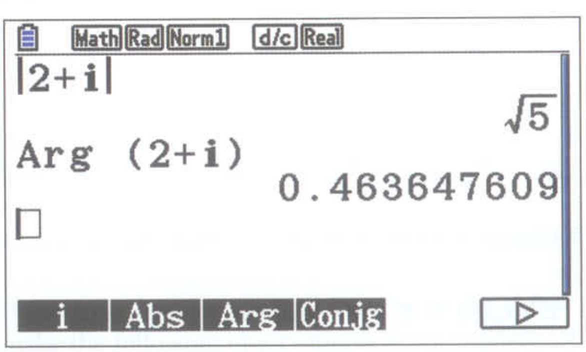
## Using a calculator

On TI models, remember to use Menu 2, 9 to access the complex number capabilities.




Use run mode if using Casio.

Press the Option key (OPTN) followed by F3 - complex,




**Example A.7.13**

If $z = 1 + 2i$ and $w = x - i$, find:

a $|z + 4|$ b $|z + w|$

a First, we need to determine the complex number $z + 4$

$z + 4 = (1 + 2i) + 4 = 5 + 2i$.

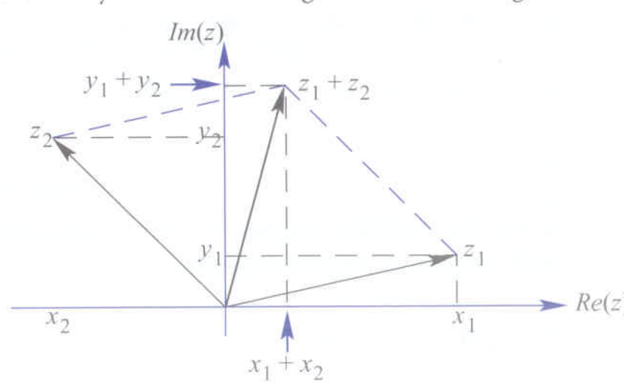
Then we have, $|5 + 2i| = \sqrt{25 + 4} = \sqrt{29}$

b First, we need to determine the complex number $z + w$:

$$
\begin{aligned}
z+w &= (1+2i)+(x-i) \\
&= (x+1)+i \\
&= |(x+1)+i| \\
&= \sqrt{(x+1)^2+1} \\
&= \sqrt{x^2+2x+2}
\end{aligned}
$$

## Adding complex numbers - geometric representation

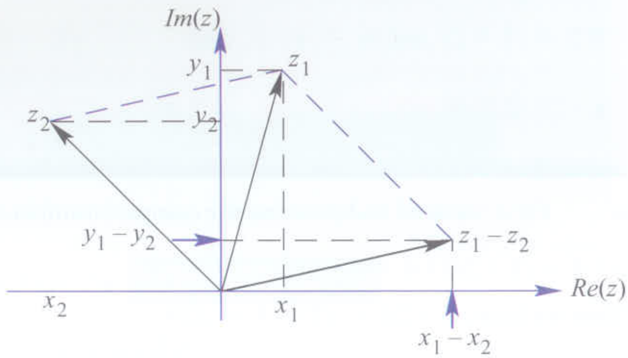
The addition of two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ can be considered in the same way as the addition of two vectors. That is, if $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are represented by directed line segments from the origin $0 + 0i$ their sum, $(x_1 + x_2) + (y_1 + y_2)i$ can also be represented by a directed line segment from the origin $0 + 0i$.




e.g. if $z_1 = 6 + 2i$ and $z_2 = -4 + 4i$ then $z_1 + z_2 = 2 + 6i$

## Subtracting complex numbers - geometric representation

Subtracting two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ can be considered in the same way as subtracting two vectors. That is, if $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are represented by directed line segments from the origin $0 + 0i$. Subtracting $z_2$ from $z_1$, i.e. $z_1 - z_2$ we obtain $(x_1 - x_2) + (y_1 - y_2)i$ which can also be represented by a directed line segment from the origin $0 + 0i$.




e.g. if $z_1 = 2 + 6i$ and $z_2 = -4 + 4i$ then $z_1 - z_2 = 6 + 2i$

The similarities between complex numbers and vectors in two dimensions make much of the theory interchangeable. Often, complex numbers are represented by the same notation as used in vector theory. For example, if the point P on the Argand diagram represents the complex number $z = 2 + 3i$ then the vector $\overrightarrow{OP} = [2, 3]$ would represent the same point. However, at this stage we will concentrate on features that deal directly with the complex numbers field.

## Exercise A.7.2

1. Show the following complex numbers on an Argand diagram:

   a $2 + i$ b $-6i$

   c $4 - 3i$ d $2(1 - i)$

   e $-3(1 - i)$ f $(1 + 2i)^2$

2. a For the complex number $z = 1 + i$, represent the following on an Argand diagram:

   i $zi$ ii $zi^2$ iii $zi^3$

   iv $zi^4$

   b What is the geometrical effect of multiplying a complex number by $i$?

   i $z^*$ ii $z + z^*$ iii $z - z^*$

   Describe the geometrical significance of each of the operations in part b.

3. If $z_1 = 1 + 2i$ and $z_2 = 1 + i$, show each of the following on an Argand diagram:

   a $z_1^2$ b $\frac{1}{z_2}$ c $z_1 z_2$

   d $2z_1 - z_2$ e $\overline{z_1 z_2}$ f $\overline{z_1} + \overline{z_2}$

4. Find the modulus and argument of:

   a $1 + \sqrt{3}i$ b $1 - \sqrt{3}i$ c $1 + \sqrt{2}i$

5. Consider the two complex numbers $z = a + bi$ and $w = -a + bi$.

   Find $|z|, |w|, |zw|$.

   Find: i $Arg(z + w)$ ii $Arg(z - w)$.

6. If $z = (x - 3) + i(x + 3)$, find: a $|z|$ b $\{x : |z| = 6\}$

7. If $z = 2 + i$ and $w = -1 - i$, verify the following.

   a $|z|^2 = zz^*$ b $|zw| = |z||w|$

   c $|w^3| = |w|^3$ d $|z + w| \le |z| + |w|$

   e $Arg(zw) = Arg(z) + Arg(w)$

   What is the geometrical significance of part d?

8. If $w = \frac{z + 1}{z - 1}$ and $|z| = 1$, find $Re(w)$.

9. Given that $|w| = 5$, find

   a $|-3w|$ b $|\overline{w}|$ c $|2iw|$.

10. If $\text{Arg}(z) = 0$, show that $z$ is real and positive.

11. A complex number $w$ is such that $w$ is purely imaginary.

Show that $\text{Arg}(w) = \pm\frac{\pi}{2}$.

12. a If $\arg(z) = \frac{\pi}{6}$ and $z = x + iy$, show that $\sqrt{3}y = x$.

b Find $z$ if $|z-1| = 1$ and $\arg(z-i) = 0$.

13. a If the complex number $z$ satisfies the equations:

$$\arg(z+1) = \frac{\pi}{6} \text{ and } \arg(z-1) = \frac{2\pi}{3},$$

show that $z = \frac{1}{2}(1+\sqrt{3}i)$.

b If $w$ and $z$ are two complex numbers such that $|z-w| = |z+w|$,

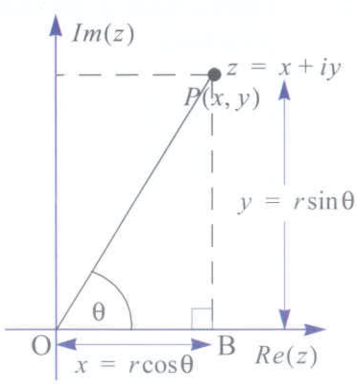
show that $|\arg(z) - \arg(w)| = \frac{\pi}{2}$ or $\frac{3\pi}{2}$.

Extra questions

## Polar Form

So far we have been dealing with complex numbers of the form $z = x + iy$, where $x$ and $y$ are real numbers. Such a representation of a complex number is known as a **rectangular** representation.

However, the position of a complex number on an Argand diagram has also been described by its magnitude (i.e. its modulus) and the angle which it makes with the positive $Re(z)$-axis. When we represent a complex number by making use of its modulus and argument, we say that the complex number is in **polar form**.




To **convert from the rectangular form to the polar form**, we make the following observations: From triangle OBP, we have:

1. $\sin(\theta) = \frac{BP}{OP} = \frac{y}{r} \Rightarrow y = r\sin(\theta)$

2. $\cos(\theta) = \frac{OB}{OP} = \frac{x}{r} \Rightarrow x = r\cos(\theta)$

Therefore, we can rewrite the complex number $z$ as follows:

$z = x + iy = r\cos(\theta) + ir\sin(\theta)$

$= r(\cos\theta + i\sin\theta)$ - we say that **z is in polar form.**

Often, we abbreviate the expression $z = r(\cos\theta + i\sin\theta)$ to:

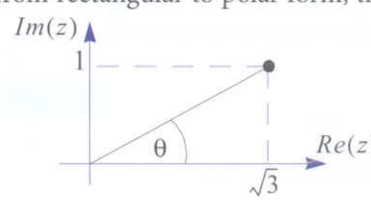
$$z = \underbrace{r}_{r}(\underbrace{c}_{c}\text{os}\theta + \underbrace{is}_{is}\text{in}\underbrace{q}_{\theta}) = rcis(\theta)$$

**Example A.7.14**

Express the following complex numbers in polar form.

a $z = \sqrt{3} + i$ b $z = -1 - i$

a When converting from rectangular to polar form, the angle $\theta$ refers to the Principal argument.




It is advisable to draw a diagram when converting from rectangular to polar form.

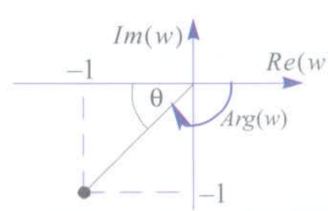
**Step 1** $\tan\theta = \frac{1}{\sqrt{3}} \Rightarrow \theta = \frac{\pi}{6}$ **Step 2** $r = \sqrt{1^2 + (\sqrt{3})^3} = 2$

Therefore, $z = \sqrt{3} + i = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = 2cis\left(\frac{\pi}{6}\right)$

b

**Step 1** $\tan\theta = \frac{1}{1} \Rightarrow \theta = \frac{\pi}{4}$

**Step 2** $r = \sqrt{1^2 + (1)^2} = \sqrt{2}$




$z = -1 - i = \sqrt{2}\left(\cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)\right) = \sqrt{2}cis\left(-\frac{3\pi}{4}\right)$

**Example A.7.15**

Convert $\sqrt{2}cis\left(\frac{3\pi}{4}\right)$ to Cartesian form.

Let $z = \sqrt{2}cis\left(\frac{3\pi}{4}\right)$. Therefore, we have:

$z = \sqrt{2}\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right)$ ('expanding' cis-term)

$= \sqrt{2}\left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)$

$= -1 + i$

## Euler Form

A third form of writing complex numbers is known as the exponential (or Euler) form. This also utilises the modulus and argument of a complex number. This is based on the rather surprising relation:

$$e^{i\theta} = cis\theta$$

Thus, for example:

Cartesian Form: $1 + i$.

Modulus = $\sqrt{2}$ and argument = $\frac{\pi}{4}$

Polar Form: $\sqrt{2}cis\frac{\pi}{4}$

Euler Form: $\sqrt{2}e^{\frac{i\pi}{4}}$

In the last chapter of this book, we will cover polynomial series and, in particular, these results:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + ...$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ...$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ...$$

Note that the 'odd function' sine consists of all the odd powers of $x$ and the 'even function' cosine consists of all the even powers of $x$.

These three series are connected:

$$e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \frac{(ix)^6}{6!} + \frac{(ix)^7}{7!} + ...$$

$$= 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \frac{x^6}{6!} - \frac{ix^7}{7!} + ...$$

The red terms correspond to the series for $\cos x$ and the green terms correspond to the series for $i \times \sin x$.

This suggests Euler's result: $e^{ix} = \cos x + i\sin x$ .

There is one case of this result (with $x = \pi$) that comes top of most lists of 'The most satisfying formulas':

$$e^{i\pi} = -1$$

This is because it encapsulates four of the most interesting numbers ($e$, $i$, $\pi$ and $-1$) in a very brief statement.

### Exercise A.7.3

1. Express each of the following complex numbers in polar form.

a $1 + i$ b $-1 + i$ c $-1 - i$

2. Express each of the following complex numbers in polar form.

a $2 + 2i$ b $\sqrt{3} + i$ c $4 - 4i$

d $3 + 4i$ e $-2 + i$ f $-2 - 3i$

g $-\sqrt{3} + i$ h $\frac{1}{2} - \frac{\sqrt{3}}{2}i$ i $3 - i$

3. Express each of the following complex numbers in Cartesian form.

a $2cis\left(\frac{\pi}{2}\right)$ b $3cis\left(\frac{\pi}{6}\right)$

c $\sqrt{2}cis\left(-\frac{\pi}{4}\right)$ d $5cis\left(\frac{3\pi}{2}\right)$

e $-8cis\left(-\frac{\pi}{3}\right)$ f $\frac{\sqrt{2}}{3}cis\left(\frac{7\pi}{3}\right)$

4. Simplify the following.

a $\frac{|2+i|}{|1-\sqrt{2}i|}$ b $\frac{zz^*}{|z|^2}$

c $Arg(z) + Arg(z^*)$

5. If $z = \sqrt{2}cis\left(\frac{\pi}{4}\right)$ and $w = 1 + \sqrt{3}i$,

find the following, giving your answer in the form $u + iv$.

a $w^*$ b $z^*$ c $wz$

6. a If $z = x + iy$, show that $z + \frac{|z|^2}{z} = 2Re(z)$.

b If $z = x + iy$, show that:

i $|z| = |\bar{z}|$ ii $z\bar{z} = |z|^2$

7. If $z = 1 + i$ and $w = -1 + \sqrt{3}i$, find:

a $|z|$ b $|w|$ c $|zw|$

d $Arg(z)$ e $Arg(w)$ f $Arg(zw)$

8. Use the laws of indices to express $e^{iA} \times e^{iB}$ as a single term.

Expand $(\cos A + i\sin A) \times (\cos B + i\sin B)$

Hence prove that:

a $\cos(A+B) = \cos A\cos B - \sin A\sin B$

b $\sin(A+B) = \sin A\cos B + \cos A\sin B$

9. By considering $(e^{ix})^n$ prove that:

$(\cos x + i\sin x)^n = \cos nx + i\sin nx$.

## de Moivre's Theorem

We begin with an important result:

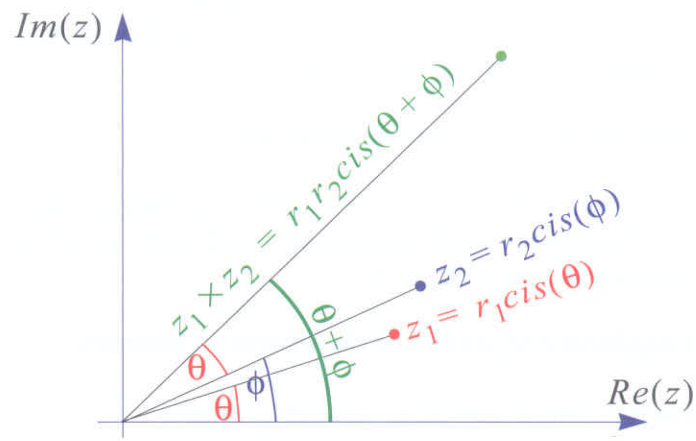
When complex numbers are expressed in polar form, their product can be found by:

1. Multiplying their moduli.

2. Adding their arguments.

Algebraically, this is: If $z_1 = r_1cis(\theta)$ and $z_2 = r_2cis(\phi)$, then $z_1 \times z_2 = r_1r_2cis(\theta + \phi)$.

Graphically, this becomes:




The powers of a complex number are a special case of this property.

Next, if $z_1 = z_2 = z = rcis(\theta)$, we then have that:

$z^2 = z \times z = rcis(\theta) \times rcis(\theta) = r^2cis(\theta + \theta)$.

That is, $z^2 = r^2cis(2\theta)$.

and: $z^3 = z \times z^2 = rcis(\theta) \times r^2cis(2\theta) = r^3cis(\theta + 2\theta)$.

That is, $z^3 = r^3cis(3\theta)$.

In general then, we have that $z^n = r^ncis(n\theta)$.

**de Moivre's Theorem states:**

$$(r(\cos\theta + i\sin\theta))^n = r^n((\cos n\theta + i\sin n\theta))$$

**Proof:** (By mathematical induction)

Let $P(n)$ be the proposition that $(rcis(\theta))^n = r^ncis(n\theta)$.

For $n = 1$, we have that

L.H.S $= (rcis(\theta))^1 = rcis(\theta) = r^1 cis(1 \times \theta) =$ R.H.S

Therefore, $P(n)$ is true for $n = 1$.

Assume now that $P(n)$ is true for $n = k$,

that is, $(rcis(\theta))^k = r^k cis(k\theta)$.

Then, for $n = k + 1$, we have

$$\begin{aligned}(rcis(\theta))^{k+1} &= (rcis(\theta))^k (rcis(\theta)) \\ &= r^k cis(k\theta)(rcis(\theta)) \\ &= r^{k+1} cis(k\theta) cis(\theta) \\ &= r^{k+1} cis(k\theta + \theta) \\ &= r^{k+1} cis((k+1)\theta)\end{aligned}$$

Therefore, we have that $P(k+1)$ is true whenever $P(k)$ is true. Therefore, as $P(1)$ is true, by the Principle of Mathematical Induction, $P(n)$ is true for $n = 1, 2, 3, \ldots$

Note that the case $n = 0$ is the trivial case.

Notice that de Moivre's Theorem holds for all integral values of $n$, both positive and negative, i.e. $n \in \mathbb{Z} \cup \{0\}$ as well as rational values of $n$, i.e. $n \in \mathbb{Q}$.

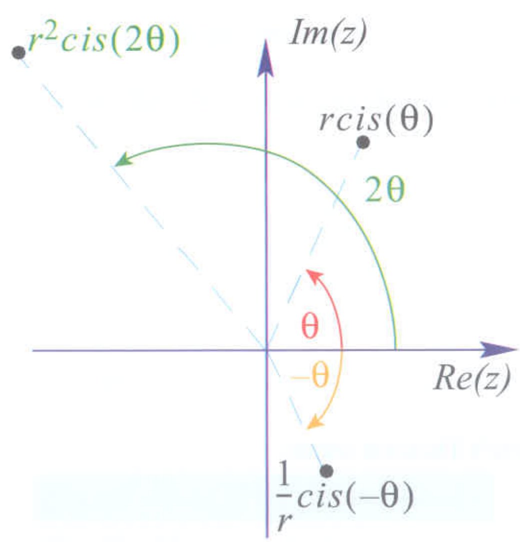
## Graphical properties of de Moivre's Theorem

For the complex number $z = rcis(\theta)$, we have

1. $z^{-1} = (rcis(\theta))^{-1} = \frac{1}{r} cis(-\theta)$

   i.e. $|z^{-1}| = |z|^{-1} = \frac{1}{r}$ and $\arg(z^{-1}) = -\theta$.

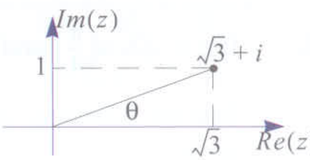
2. $z^2 = (rcis(\theta))^2 = r^2 cis(2\theta)$




**Example A.7.16**

Find $(\sqrt{3} + i)^5$ using de Moivre's Theorem.

Let $z = \sqrt{3} + i$.




This means:

$r = |\sqrt{3} + i| = \sqrt{(\sqrt{3})^2 + 1^2} = 2$ and $\theta = Tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$.

Therefore, we have that $z = \sqrt{3} + i = 2cis\left(\frac{\pi}{6}\right)$.

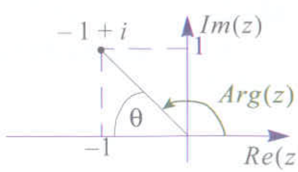
Using de Moivre's Theorem, we have:

$$\begin{aligned}(\sqrt{3} + i)^5 &= 2^5 cis\left(\frac{5\pi}{6}\right) = 32 cis\left(\frac{5\pi}{6}\right) \\ &= 32\left(\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\right) \\ &= 32\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) \\ &= -16\sqrt{3} + 16i\end{aligned}$$

**Example A.7.17**

Find $(-1 + i)^{-4}$ using de Moivre's Theorem.

Let $z = -1 + i$.




This means:

$r = |-1 + i| = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$ and

$\theta = Tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4} \therefore Arg(z) = \frac{3\pi}{4}$.

Therefore, we have that $z = -1 + i = \sqrt{2} cis\left(\frac{3\pi}{4}\right)$.

Using de Moivre's Theorem, we have:

$$(-1+i)^{-4} = (\sqrt{2})^{-4}cis\left(-4\times\frac{3\pi}{4}\right) = \frac{1}{(\sqrt{2})^4}cis(-3\pi)$$

$$= \frac{1}{4}(\cos(-3\pi)+i\sin(-3\pi)) \quad = \frac{1}{4}(-1+0i) \quad = -\frac{1}{4}$$

**Example A.7.18**

Express $\frac{1+i}{(1-i)^3}$ in polar form.

We first convert both numerator and denominator into polar form.

$$1+i = \sqrt{2}cis\left(\frac{\pi}{4}\right) \text{ [standard result]}$$

and $1-i = \sqrt{2}cis\left(-\frac{\pi}{4}\right) \therefore (1-i)^3 = (\sqrt{2})^3 cis\left(-\frac{3\pi}{4}\right)$

Therefore,

$$\frac{1+i}{(1-i)^3} = \frac{\sqrt{2}cis\left(\frac{\pi}{4}\right)}{2\sqrt{2}cis\left(-\frac{3\pi}{4}\right)} = \frac{1}{2}cis\left[\left(\frac{\pi}{4}\right)-\left(-\frac{3\pi}{4}\right)\right] = \frac{1}{2}cis(\pi)$$

$$= \frac{1}{2}\cos\pi + \frac{1}{2}i\sin\pi.$$

**Example A.7.19**

Simplify:

a $(1+i)^5+(1-i)^5$ b $(1+i)^5(1-i)^5$

a We first convert each term into its polar form:

$$1+i = \sqrt{2}cis\left(\frac{\pi}{4}\right)$$

$$(1+i)^5 = \left(\sqrt{2}\right)^5 cis\left(\frac{5\pi}{4}\right)$$

$$= 4\sqrt{2}cis\left(\frac{5\pi}{4}\right)$$

$$1-i = \sqrt{2}cis\left(-\frac{\pi}{4}\right)$$

$$(1-i)^5 = \left(\sqrt{2}\right)^5 cis\left(-\frac{5\pi}{4}\right)$$

$$= 4\sqrt{2}cis\left(-\frac{5\pi}{4}\right)$$

It follows that:

$$= 4\sqrt{2}\left[\left(-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right)+\left(-\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}i\right)\right]$$

$$= 4\sqrt{2}\left[-\frac{2}{\sqrt{2}}\right]$$

$$= -8$$

b Using the previous results we have,

$$(1+i)^5(1-i)^5 = 4\sqrt{2}cis\left(\frac{5\pi}{4}\right)\times 4\sqrt{2}cis\left(-\frac{5\pi}{4}\right)$$

$$= 32cis\left(\frac{5\pi}{4}-\frac{5\pi}{4}\right)$$

$$= 32cis(0)$$

$$= 32$$

Notice that whenever we add or multiply the complex numbers $rcis(\theta)$ and $rcis(-\theta)$, a purely real complex number will always result. This can seen as follows:

1. Adding

$$\begin{aligned} rcis(\theta)+rcis(-\theta) &= r[cis(\theta)+cis(-\theta)] \\ &= r[(\cos\theta+i\sin\theta)+(\cos(-\theta)+i\sin(-\theta))] \\ &= r[(\cos\theta+i\sin\theta)+(\cos\theta-i\sin\theta)] \\ &= r[2\cos\theta] \\ &= 2r\cos\theta \end{aligned}$$

2. Multiplying

$$\begin{aligned} rcis(\theta)\times rcis(-\theta) &= r^2[cis(\theta)\times cis(-\theta)] \\ &= r^2[cis(\theta-\theta)] \\ &= r^2cis(0) \\ &= r^2 \end{aligned}$$

## Exercise A.7.4

1. Express each of the following in the form $x + iy$.

| | | | |
|---|---|---|---|
| a | $(1+i)^5$ | b | $(-1+i)^4$ |
| c | $(2+2i)^3$ | d | $(-\sqrt{3}+i)^4$ |
| e | $(\sqrt{3}-i)^5$ | f | $(3-4i)^3$ |

2. Express each of the following in the form $x + iy$.

| | | | |
|---|---|---|---|
| a | $(1+i)^{-5}$ | b | $(-1+i)^{-4}$ |
| c | $(2+2i)^{-3}$ | d | $(-\sqrt{3}+i)^{-4}$ |
| e | $(\sqrt{3}-i)^{-5}$ | f | $(3-4i)^{-3}$ |

3. Express each of the following in the form $x + iy$.

| | | | |
|---|---|---|---|
| a | $\left(2cis\left(\frac{\pi}{2}\right)\right)^3$ | b | $\left(3cis\left(\frac{\pi}{6}\right)\right)^4$ |
| c | $\left(\sqrt{2}cis\left(-\frac{\pi}{4}\right)\right)^{-2}$ | d | $\left(5cis\left(\frac{3\pi}{2}\right)\right)^{-3}$ |
| e | $\left(-8cis\left(-\frac{\pi}{3}\right)\right)^{-1}$ | f | $\left(\frac{\sqrt{2}}{3}cis\left(\frac{7\pi}{3}\right)\right)^4$ |

4. Find each of the following, expressing your answer in the form $x + iy$.

a $(1+i)^3(2-2i)^4$

b $(\sqrt{3}+i)^2(1-i)^2$

c $\dfrac{(2+2\sqrt{3}i)^3}{(i-1)^2}$

d $(\sqrt{3}+i)^4(1+\sqrt{3}i)^4$

e $\dfrac{(3+4i)^4}{(3-4i)^2}$

f $\dfrac{(1+i)^4}{(1-i)^2}$

5. a Prove that $cis(\theta + 2k\pi) = cis(\theta)$, for all integer values of $k$.

Using part a, evaluate the following.

i $cis(37\pi)$ ii $cis(-43\pi)$

iii $cis\left(\frac{29}{2}\pi\right)$.

6. Simplify the following.

a $cis(\pi)cis\left(-\frac{3\pi}{2}\right)$

b $2cis\left(\frac{\pi}{12}\right) \times 6cis\left(\frac{\pi}{6}\right)$

c $\dfrac{\sqrt{8}cis\left(\frac{\pi}{8}\right)}{\sqrt{2}cis\left(-\frac{\pi}{2}\right)}$

7. a Express $cis\left(\frac{\pi}{4}\right)$ and $cis\left(\frac{\pi}{3}\right)$ in the form $x + iy$.

Hence, express $cis\left(\frac{7\pi}{12}\right)$ in the form $x + iy$.

b Use part a to find the exact value of:

i $\sin\left(\frac{7\pi}{12}\right)$ ii $\cos\left(\frac{7\pi}{12}\right)$

8. Use De Moivre's theorem to prove that:

if $z = rcis(\theta)$ then $(\bar{z})^n = \overline{(z^n)}$.

Extra questions

## The *n*th roots of a Complex Number

**Definition** The $n$th roots of the complex number $x + iy$ are the solutions of the equation $z^n = x + iy$.

de Moivre's Theorem suggests a geometric approach.

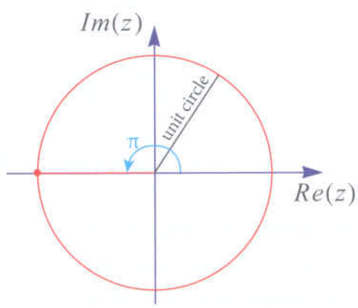
**Example A.7.20**

Find the cube roots of −1.

This question amounts to asking for all the solutions to:

$z^3 = -1$ or $z^3 = \text{cis}\pi$.

First locate −1 on the Argand Diagram:




As a consequence of de Moivre's Theorem, any solution to this question must have a modulus of 1 ($1^3 = 1$).

Also, any solution must have an argument which, when multiplied by 3, will give π.

The most obvious answer is an argument of $\frac{\pi}{3}$.

Is $\text{cis}\frac{\pi}{3} = \frac{1}{2} + \frac{\sqrt{3}}{2}i$ a cube root of −1?

Check: $\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^3 = \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$

$$= \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\left(\frac{1}{4} + \frac{\sqrt{3}}{4}i + \frac{\sqrt{3}}{4}i + \frac{3}{4}i^2\right)$$

$$= \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\left(\frac{1}{4} + \frac{\sqrt{3}}{2}i - \frac{3}{4}\right)$$

$$= \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$$

$$= -\frac{1}{4} + \frac{\sqrt{3}}{4}i - \frac{\sqrt{3}}{4}i + \frac{3}{4}i^2$$

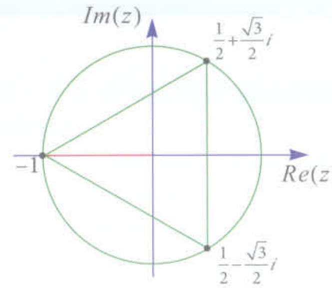
$$= -\frac{1}{4} - \frac{3}{4}$$

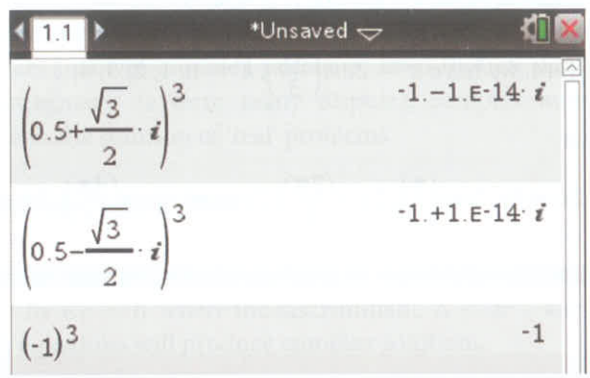
$$= -1$$

However, there are two other answers:

$\text{cis}-\pi = -1$ and $\text{cis}-\frac{\pi}{3} = \frac{1}{2} - \frac{\sqrt{3}}{2}i$

The three solutions lie at the vertices of an equilateral triangle:




Geometrically, we have that the $n$th roots of a complex number are represented in an Argand diagram as the vertices of a regular polygon of $n$ sides, inscribed in a circle of radius $\sqrt[n]{r}$, and spaced at intervals of $\frac{2\pi}{n}$ from each other.

The steps involved in solving equations of the form $z^n = x + iy$ (even for the case that $y = 0$) are:

Step 1. Express $x + iy$ in polar form, $r\,cis(\theta)$

Step 2. Realise that $r\,cis(\theta) = r\,cis(\theta + 2k\pi)$, where $k$ is an integer, because every time you add another $2\pi$, you return to the same position.

Step 3. Use de Moivre's theorem:

$$z^n = r\,cis(\theta + 2k\pi) \therefore z = [r\,cis(\theta + 2k\pi)]^{\frac{1}{n}} = r^{\frac{1}{n}}\,cis\left(\frac{\theta + 2k\pi}{n}\right)$$

Step 4. Use $n$ values of $k$, usually start at $k = 0, 1, \ldots$ and end at $k = n-1$. This will produce the $n$ required solutions

**Example A.7.21**

Find the 6th roots of 64, leaving your answer in polar form.

Setting $z^6 = 64$ we have,

$$z^6 = 64 + 0i = 64[cis(0) + i\sin(0)]$$
$$= 64cis(0)$$
$$= 64cis(0 + 2k\pi)$$

$$\therefore z = 64^{1/6}cis\left(\frac{2k\pi}{6}\right), k = 0, 1, 2, 3, 4, 5$$

Therefore, we have $z = 2cis\left(\frac{\pi k}{3}\right), k = 0, 1, 2, 3, 4, 5$.

So that,

$$z = 2cis(0), 2cis\left(\frac{\pi}{3}\right), 2cis\left(\frac{2\pi}{3}\right), 2cis(\pi), 2cis\left(\frac{4\pi}{3}\right), 2cis\left(\frac{5\pi}{3}\right)$$

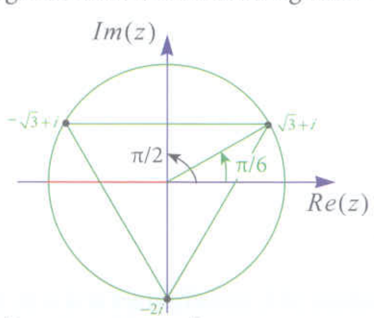
**Example A.7.22**

Find the cube roots of $8i$.

$8i$ has a modulus of 8 and an argument of $\frac{\pi}{2}$.

By de Moivre's Theorem, one these cube roots will have an argument of one third of the argument of $8i$. The moduli of all the roots will be the cube root of 8 (= 2).

The other two roots will be at the vertices of an equilateral triangle (triangle because we are looking for a cube root).




That is, $z = \sqrt{3} + i$ or $z = -\sqrt{3} + i$ or $z = -2i$.

**Example A.7.23**

Find the four fourth roots of $1 + i\sqrt{3}$. Give your answer in polar form.

We start by expressing $1 + i\sqrt{3}$ in its polar form:

$$1 + i\sqrt{3} = 2cis\left(\frac{\pi}{3}\right).$$

Then, set $z^4 = 2cis\left(\frac{\pi}{3}\right) = 2cis\left(\frac{\pi}{3} + 2k\pi\right) = 2cis\left(\frac{\pi + 6k\pi}{3}\right)$

So that, $z = \sqrt[4]{2}cis\left(\frac{\pi + 6k\pi}{12}\right), k = 0, 1, 2, 3$.

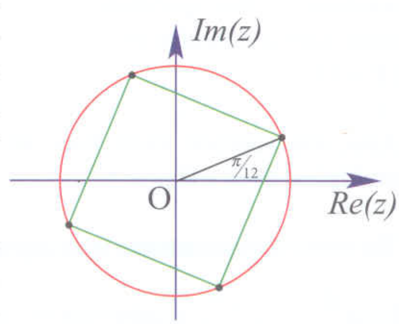
For $k = 0$, $z = \sqrt[4]{2}cis\left(\frac{\pi}{12}\right)$;

$k = 1$, $z = \sqrt[4]{2}cis\left(\frac{\pi + 6\pi}{12}\right) = \sqrt[4]{2}cis\left(\frac{7\pi}{12}\right)$;

$k = 2$, $z = \sqrt[4]{2}cis\left(\frac{\pi + 12\pi}{12}\right) = \sqrt[4]{2}cis\left(\frac{13\pi}{12}\right)$;

$k = 3$, $z = \sqrt[4]{2}cis\left(\frac{\pi + 18\pi}{12}\right) = \sqrt[4]{2}cis\left(\frac{19\pi}{12}\right)$

Therefore, the four roots of $1 + i\sqrt{3}$ lie on the circumference of a circle of radius $\sqrt[4]{2}$ units and are evenly separated by an angle of $\frac{\pi}{2}$.




Again notice that the roots in this instance do not occur in conjugate pairs.

### Exercise A.7.5

1. Use the $n$th root method to solve the following:

   a $z^3 = 27$  b $z^3 = 27i$

   c $z^3 = -8i$  d $z^4 = -16$

2. Find the fourth roots of $-4$ in the form $x + iy$ and hence factorise $z^4 + 4$ into linear factors.

3. Find the square roots of:

a $i$ b $3+4i$ c $-1+\sqrt{3}i$.

Represent these roots on an Argand diagram.

4. Find the cube roots of:

a $1-i$ b $-1+\sqrt{3}i$ c $i$

Represent these roots on an Argand diagram.

5. Solve the following equations.

a $z^4 = 1+i$

b $z^4 = i$

c $z^3 + i = 0$

d $z^4 = 8-8\sqrt{3}i$

e $z^3 = 64i$

f $z^2 = \sqrt{3}+i$

Represent these roots on an Argand diagram.

6 a Find the cube root of unity.

b Hence, show that if $w^3 = 1$, then $1+w+w^2 = 0$.

7. Three points, of which $1+i\sqrt{3}$ is one point, lie on the circumference of a circle of radius 2 units and centre at the origin. If these three points form the vertices of an equilateral triangle, find the other two points.

Extra questions

# Polynomials

This section will look at polynomials with real coefficients in which the variable may take complex values. To emphasise this, the variable is generally labelled $z$ (rather than $x$).

$P(z) = 2z^2 + 3z - 4$ is an example of a complex polynomial with real coefficients.

$P(z) = 2z^2 + 3iz - 4$ is an example of a complex polynomial with a complex coefficient (shown in green). Such polynomials are not included in this course.

Arising from such polynomials are equations with complex solutions. Our cover shows a 'fractal' - a form that mimics many natural objects. Solving many problems in the natural sciences involves complex numbers. Even though they may be 'imaginary' (a term many dispute), complex numbers figure in the solution of 'real' problems.

## Quadratic equations

We start this section by looking at equations of the form $ax^2+bx+c = 0$ where the discriminant, $\Delta = b^2-4ac < 0$. Such equations will produce complex solutions.

**Example A.7.24**

Factorise, over the complex number field, $z^2+2z+2$. Hence, solve the equation $z^2+2z+2=0$.

We start the same way we would when dealing with any quadratic expression:

$$\begin{aligned} z^2+2z+2 &= (z^2+2z+1)+1 \text{ (complete the square)} \\ &= (z+1)^2+1 \\ &= (z+1)^2-i^2 \text{ (difference of two squares)} \\ &= (z+1+i)(z+1-i) \end{aligned}$$

To solve $z^2+2z+2=0$, we have:

$(z+1+i)(z+1-i) = 0 \Leftrightarrow z = -1-i$ or $z = -1+i$.

Therefore, the two complex solutions are $z = -1 - i$ and $z = -1 + i$. Notice that the solutions are a conjugate pair.

**Example A.7.25**

Solve the equation $z^2+3z+5=0$ over the complex field.

Rather than factorizing the equation, we will use the quadratic formula.

$$z^2+3z+5 = 0 \Leftrightarrow z = \frac{-3\pm\sqrt{3^2-4\times1\times5}}{2\times1}$$
$$= \frac{-3\pm\sqrt{-11}}{2}$$
$$= \frac{-3\pm\sqrt{11}i}{2}$$

Therefore, the two complex solutions are

$$z = -\frac{3}{2}+\frac{\sqrt{11}}{2}i, z = -\frac{3}{2}-\frac{\sqrt{11}}{2}i.$$

Again, notice the conjugate pair that make up the solution.

Quadratics also come in a 'hidden form'. For example, the equation $z^6+4z^3-5 = 0$ can be considered to be a 'hidden form'. i.e. letting $w = z^3$ we have $w^2+4w-5 = 0$. And so we can then solve the quadratic in *w*. We could then obtain solutions for *z*.

**Example A.7.26**

Solve the equation $z^4+4z^2-5=0$ over the complex field.

Let $w = z^2$ so that the equation $z^4+4z^2-5 = 0$ is transformed into the quadratic $w^2+4w-5 = 0$.

Then, we have $w^2+4w-5 = 0 \Leftrightarrow (w+5)(w-1) = 0$

$$\Leftrightarrow (z^2+5)(z^2-1) = 0$$

$$\Leftrightarrow (z-\sqrt{5}i)(z+\sqrt{5}i)(z-1)(z+1) = 0$$

Therefore, we have that $z = \sqrt{5}i$ or $z = -\sqrt{5}i$ or $z = 1$ or $z = -1$.

That is, we have four solutions, two real and two complex (again, the complex solutions are conjugate pairs).

## Exercise A.7.6

1. Factorise the following over the complex number field.

| | | | |
|---|---|---|---|
| a | $x^2-6x+10$ | b | $x^2+4x+13$ |
| c | $x^2-2x+2$ | d | $z^2+4z+5$ |
| e | $z^2-3z+4$ | f | $z^2+10z+30$ |
| g | $4w^2+4w+17$ | h | $3w^2-6w+6$ |
| i | $-2w^2+8w-11$ | | |

2. Solve the following over the complex number field.

a $z^2+4z+8 = 0$

b $z^2-z+3 = 0$

c $3z^2-3z+1 = 0$

d $2w^2+5w+4 = 0$

e $w^2+10w+29 = 0$

3. Solve the following over the complex number field.

a $z^4-3z^2-4 = 0$

b $w^4-8w^2-9 = 0$

c $z^4-5z^2-36 = 0$

4. Factorise the following over the complex number field.

| | | | |
|---|---|---|---|
| a | $z^2+25$ | b | $z^2+49$ |
| c | $z^2+4z+5$ | d | $z^2+6z+11$ |
| e | $z^4+2z^2-8$ | f | $z^4-z^2-6$ |

## Polynomial equations (of order ≥ 3)

We now look at some of the more general polynomial equations that provide a combination of real and imaginary roots and factors. The important thing to remember is that the laws for real polynomials hold equally well for complex polynomials.

A polynomial, $P(z)$ of degree $n$ in one variable is an expression of the form

$$a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0$$

If the coefficients, $a_n, a_{n-1}, \ldots, a_1, a_0$ are real, the polynomial is a **polynomial over the real number field**, while if they are complex numbers, the polynomial is a **polynomial over the complex field**. We shall, however, concentrate on polynomials over the real field.

We state some standard results:

### Remainder Theorem

If a polynomial $P(x)$ is divided by a linear polynomial $(x-a)$, the remainder is $P(a)$.

### Factor Theorem

If, when a polynomial $P(x)$ is divided by a linear polynomial $(x-a)$, the remainder $P(a)$ is zero, then $(x-a)$ is a factor of $P(x)$.

### Fundamental Theorem of Algebra

Every polynomial equation of the form $P(z) = 0$, $z \in C$, of degree $n \in \mathbb{Q}^+$ has at least one complex root.

This theorem is the basis for the next important result:

A polynomial $P_n(z)$, $z \in C$, of degree $n \in \mathbb{Q}^+$, can be expressed as the product of n linear factors and hence, produce exactly n solutions to the equation $P_n(z) = 0$.

We have already observed, in previous examples, the occurrence of conjugate pairs when solving quadratics with real coefficients. We now state another result.

### Conjugate Root Theorem (C.R.T)

The complex roots of a polynomial equation with real coefficients occur in conjugate pairs.

**Example A.7.27**

Factorise the polynomial $z^3-3z^2+4z-12$, hence solve $z^3-3z^2+4z-12=0$.

Grouping like terms, we have:

$z^3-3z^2+4z-12 = z^2(z-3)+4(z-3) = (z^2+4)(z-3)$

i.e. $z^3-3z^2+4z-12 = (z-2i)(z+2i)(z-3)$

And so, $z^3-3z^2+4z-12 = 0 \Leftrightarrow (z-2i)(z+2i)(z-3) = 0$

Therefore, we have that $z = 2i$ or $z = -2i$ or $z = 3$.

We observe that two of the roots are conjugate pairs, and when we look at the polynomial, we see that all of the coefficients are real (as expected from the C.R.T).

**Example A.7.28**

Given that $z = 1-i$ is a root of the equation $2z^3-7z^2+10z-6 = 0$, find the other roots.

As all of the coefficients of the polynomial are real, it means that the C.R.T applies. That is, given that $z = 1-i$ is a root, so too then, is $z = 1+i$.

Therefore, we have two factors, namely, $z-1+i$ and $z-1-i$.

This means that $(z-1+i)(z-1-i) = z^2-2z+2$ is also a factor.

As in the last example, we can factorise by inspection: $2z^3-7z^2+10z-6 = (az+b)(z^2-2z+2)$

That is, knowing that we are looking for a cubic, and given that we already have a quadratic factor, we are left with a linear factor, which is $(az+b)$. Then, comparing the coefficients of the $z^3$ term and the constant term we have that:

$a = 2$ and $2b = -6 \Leftrightarrow b = -3$.

That is, $2z^3-7z^2+10z-6 = (2z-3)(z^2-2z+2)$

Therefore, the roots are $1-i, 1+i, \frac{3}{2}$.

**Example A.7.29**

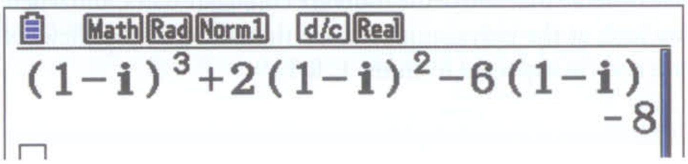
If $z-1+i$ is a factor of $P(z) = z^3+2z^2-6z+k$, find the value of $k$.

Given that $z-1+i$ is a factor of $P(z) = z^3+2z^2-6z+k$, then, by the factor theorem we must have that $P(1-i) = 0$.

So, $(1-i)^3+2(1-i)^2-6(1-i)+k = 0 \Leftrightarrow -8+k = 0$

$$\Leftrightarrow k = 8$$

Calculators are useful in situations that involve simple evaluation of complex numbers.




**Example A.7.30**

Solve the equation $z^3-4z^2+9z-10 = 0$ where $z$ is a complex number.

Let $P(z) = z^3-4z^2+9z-10$. Using trial and error (or at least factors of 10), we have:

$$P(1) = 1-4+9-10 = -4 \therefore (z-1) \text{ is not a factor.}$$

$$P(2) = 8-16+18-10 = 0 \Rightarrow (z-2) \text{ is a factor.}$$

Therefore, $P(z) = (z-2)(az^2+bz+c)$.

Comparing coefficients of the leading term and constant term we have:

$$a = 1 \text{ and } -2c = -10 \Leftrightarrow c = 5$$

Therefore, $P(z) = (z-2)(z^2+bz+5)$.

Then, comparing the coefficient of the $z^2$ term, we have that $b-2 = -4 \therefore b = -2$.

So, $P(z) = (z-2)(z^2-2z+5) = (z-2)[(z^2-2z+1)+4]$ (completing the square)

$$= (z-2)[(z-1)^2+4]$$

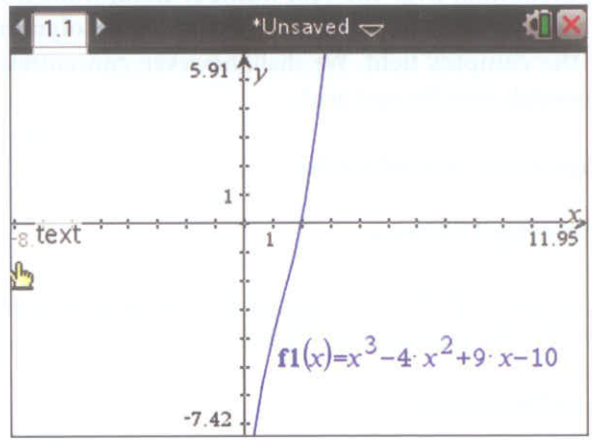
$$= (z-2)(z-1+2i)(z-1-2i)$$

Therefore, $P(z) = 0 \Leftrightarrow (z-2)(z-1+2i)(z-1-2i) = 0$

And so, $z = 2$ or $z = 1-2i$ or $z = 1+2i$.

We could have used long or synthetic division to factorize $P(z)$ to the stage $P(z) = (z-1)(z^2-3z+10)$. Both methods are equally valid.

Also, rather than using trial and error you could use your graphics calculator to help find a first real factor.




### Exercise A.7.7

1. Factorize the following over the complex number field.

   a $z^3+2z^2+z+2$ b $z^3-9z^2+z-9$

   c $z^3-2z^2+2z-4$

2. Factorise the following over the complex number field.

   a $w^3+2w-12$ b $z^3-5z^2+9z-5$

   c $z^3+z^2-2$ d $x^4-3x^2-4$

   e $w^3-2w+4$ f $z^4-625$

3. Solve each of the following over the complex number field.

   a $z^3-7z^2+31z-25 = 0$

   b $z^3-8z^2+25z-26 = 0$

   c $z^4-3z^3-2z^2+10z-12 = 0$

   d $2w^3+3w^2+2w-2 = 0$

   e $6z^4-11z^3+z^2+33z-45 = 0$

f $z^3 + 7z^2 + 16z + 10 = 0$

4. Given that $\frac{1}{2}(-1 + \sqrt{3}i)$ is a root of:

   $3z^3 + 2z^2 + 2z - 1 = 0$, find all other roots.

5. Given that $(z - 1 - 2i)$ is a factor of $2z^3 - 3z^2 + 8z + 5$ solve the equation $2z^3 - 3z^2 + 8z + 5 = 0$ over the complex number field.

6. Given that $P(2 - 3i) = 0$, find all three linear factors of $z^3 - 7z^2 + 25z - 39$.

7. Find all complex numbers, $z$, such that $z^4 - z^3 + 6z^2 - z + 15 = 0$ and $z = 1 + 2i$ is a solution to the equation.

8. Factorise the following.

   a $2z^3 - z^2 + 2z - 1$ b $z^4 + z^2 - 12$

9. Given that $2 - i$ is a root of $z^3 + az^2 + z + 5 = 0$ where $a$ is a real number, find all the roots to this equation.

10. Given that $2 + 3i$ is a root of $z^3 + az^2 + b = 0$, where $a$ and $b$ are real numbers, find all the roots of this equation.

11. Given that $2 - i$ is a root of $2z^3 - 9z^2 + 14z - 5 = 0$, find the other roots.

12. Given that $4 - i$ is a zero of:

    $P(z) = z^3 + az^2 + 33z - 34$,

    find $a$ and hence factorise $P(z)$.

13. Given that $z - 2$ and $z - 1 - i$ are factors of $P(z) = z^3 - az^2 + 6z + b$, factorise $P(z)$.

14. Solve the following over the real number field.

    a $z^6 + 7z^3 - 8 = 0$

    b $z^6 - 9z^3 + 8 = 0$

    c $z^4 - 2z^2 - 3 = 0$

    d $z^4 - 4z^2 - 5 = 0$

15. Write down an equation of the lowest possible degree with real coefficients such that its roots are:

    a $3, 2 - i$

    b $2, 1, 1 + i$

    c $1 - \sqrt{3}i, 3$

    d $1 + \sqrt{2}i, -2 + \sqrt{3}i$

16. Verify that $z = -1 + \sqrt{3}i$ is a root of the equation $z^4 - 4z^2 - 16z - 16 = 0$ and hence find the other roots.

17. Given that $z = a + ib$ is a root of $z^4 - z^3 - 6z^2 + 11z + 5 = 0$ and $Re(z) = 2$, solve the equation completely.

18. If $z^n + z^{-n} = 2\cos(n\theta)$ show that:

    $5z^4 - z^3 - 6z^2 - z + 5 = 0 \Rightarrow 10\cos^2\theta - \cos\theta - 8 = 0.$

19. Show that $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$, and

    hence show that the roots of $x(16x^4 - 20x^2 + 5) = 0$

    are 0, $\cos\left(\frac{\pi}{10}\right)$, $\cos\left(\frac{3\pi}{10}\right)$, $\cos\left(\frac{7\pi}{10}\right)$, $\cos\left(\frac{9\pi}{10}\right)$.


## Fractals

Our cover picture shows a part of a computer generated fractal.

Fractals are defined as shapes that look the same at whatever magnification we look at them.

This means that most geometric shapes are not fractal. One that is, is the straight line. This is straight however much we magnify it. By contrast, the closer we look at the circumference of a circle, the less curved it appears.

The interest in fractals is that many natural features (ferns, clouds, coastlines etc.) have fractal properties.

Until the advent of computers, it seemed that fractals were to be confined to natural forms. However, an IBM computer expert, Benoit Mandelbrot, realised in the 1970s that realistic landscapes could be created by algorithms using repetitive calculations and comparatively simple rules.

Today, when we watch an animated film, it is highly likely that the clouds, forests, mountains etc. we see are created by a computer using fractal mathematics.

Mandelbrot's early investigations were based on complex numbers. A complex number $c$ is a member of the Mandelbrot Set if:

$c, z \in \mathbb{C}$ with $z_0 = 0$ and the iterative scheme $z_{n+1} = z_n^2 + c$ leads to a sequence of complex numbers whose moduli do not tend to infinity.

For example:

If $c = 0.1i$, the sequence is:

$$z_1 = z_0^2 + 0.1i = 0^2 + 0.1i = 0.1i$$

$$z_2 = z_1^2 + 0.1i = (0.1i)^2 + 0.1i = -0.01 + 0.1i$$

$$z_3 = z_2^2 + 0.1i = (-0.01 + 0.1i)^2 + 0.1i = -0.0099 + 0.098i$$

A longer list of the sequence (rounded to 5 decimal places) is :

| Iteration (n) | Re(z) | Im(z) | Modulus |
|---|---|---|---|
| 0 | 0.00000 | 0.00000 | 0.00000 |
| 1 | 0.00000 | 0.10000 | 0.10000 |
| 2 | -0.01000 | 0.10000 | 0.10050 |
| 3 | -0.00990 | 0.09800 | 0.09850 |
| 4 | -0.00951 | 0.09806 | 0.09852 |
| 5 | -0.00953 | 0.09814 | 0.09860 |
| 6 | -0.00954 | 0.09813 | 0.09859 |

A calculator implementation of this is illustrated in this video clip:

For this value, the modulus of successive iterates settles down to a small number (~0.09859 ). Thus $0.1i$ is a member of the Mandelbrot set.

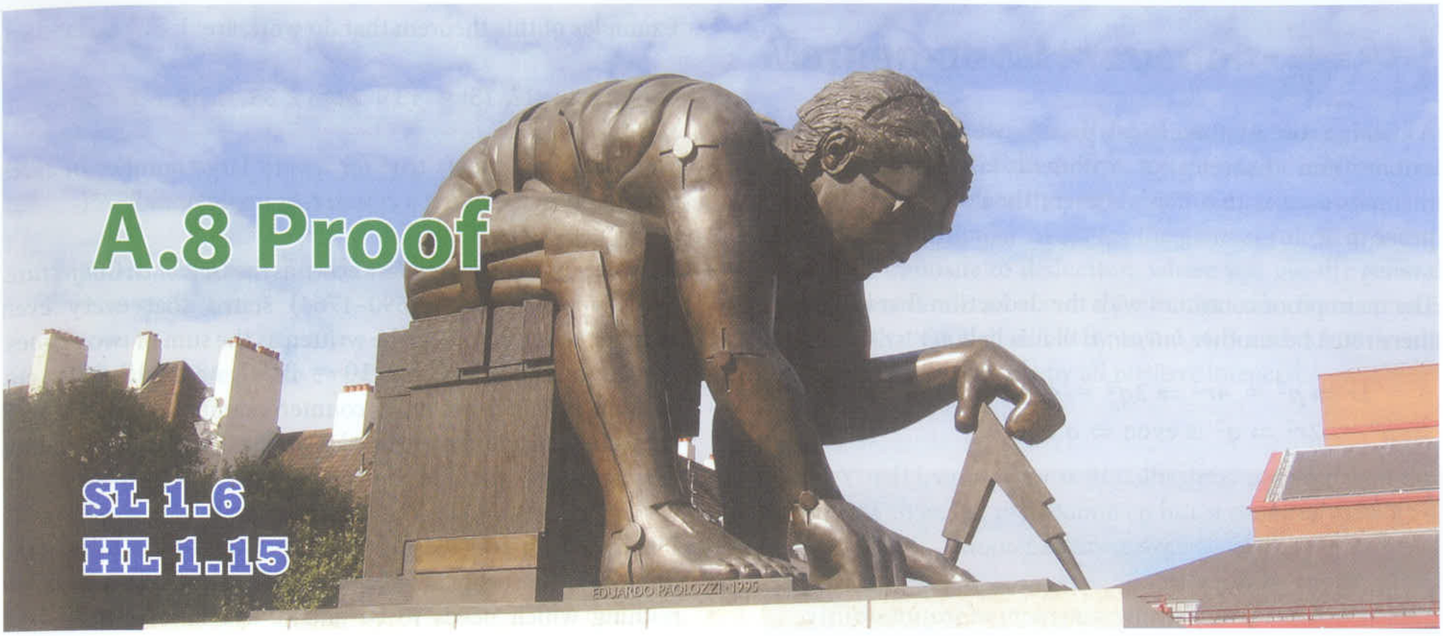
By contrast, if you use $c = 1 + 2i$, you should find that the iterates rapidly become huge. $1 + 2i$ is not a member of the Mandelbrot set. It would seem that the cut-off between the two (when shown on an Argand Diagram) should be a neat circle of radius 1. However, it is not!

As a postscript, the later work of American artist Jackson Pollock has many fractal qualities. Pollock often worked with his canvas on the floor. He then used big brushes to spatter paint onto it.

At first sight, the results look chaotic. Many people, after a longer look, begin to see natural forms such as forests and find his images restful.

Surprisingly, forgers have found it very difficult to paint successful copies of Pollock's work.

Answers




# A.8 Proof

**SL 1.6**
**HL 1.15**

## Further Methods of Proof

In the SL book, we introduced some of the most common methods of proof. We also discussed some techniques for choosing appropriate methods of proof. However, some of the great proofs of mathematics have been 'mould breakers'. Mathematicians use terms such as 'elegant' and, occasionally, 'beautiful' to describe such proofs.

For example, we revisit one of the proofs in that section, question 9 from Exercise A.8.3:

Prove that there exist irrational numbers $A$ & $B$ such that $A^B$ is rational.

This statement is very far from obviously true.

This can be answered in the affirmative by considering $\sqrt{2}^{\sqrt{2}}$. This is either rational or irrational.

If it is rational, we have our proof.

If it is irrational consider $\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \left(\sqrt{2}\right)^2 = 2$ and the result is proved.

Note that we still do not have an example of $A$ & $B$ such that $A^B$ is rational. However, we have achieved the required result!

As a second example, consider the argument presented in this video as a proof that there are only five regular polyhedra, the so called Platonic Solids.

If this fits any mould of proof, it is probably Proof by Exhaustion.

### Proof by Contradiction

This method works by assuming that the proposition is false and then proving that this assumption leads to a contradiction.

The number $\sqrt{2}$ greatly interested classical Greek mathematicians who were unable to find a number that, when it was squared, gave exactly 2.

Modern students are often fooled into thinking that their calculators give an exact square root for 2 as when 2 is entered and the square root button is pressed, a result (depending on the model of calculator) of 1.414213562 is produced. When this is squared, exactly 2 results – but not because we have an exact square root. It results from the way in which the calculator is designed to calculate with more figures than it actually displays.

| | |
|---|---|
| $\sqrt{2}$ | 1.41421 |
| $(1.4142135623731)^2$ | 2. |

The first answer is stored to more figures than are shown, the result is rounded and then displayed. The same is true of the second result which only rounds to 2. Try squaring 1.414213562, the answer is not 2.

The theorem we shall prove is that there is *no* fraction that when squared gives 2. This also implies that there is no terminating or recurring decimal that, when squared, gives exactly 2, but this further theorem requires more argument.

The method begins by assuming that there *is* a fraction $^p/_q$ ($p$ and $q$ are integers) which has been cancelled to its lowest terms, such that $^p/_q = \sqrt{2}$. From the assumption, the argument proceeds:

$$\frac{p}{q}=\sqrt{2}\Rightarrow\frac{p^2}{q^2}=2\Rightarrow p^2=2q^2\Rightarrow p^2 \text{ is even}\Rightarrow p \text{ is even}$$

As with most mathematical proofs, we have used simple axioms and theorems of arithmetic. The most complex theorem used is that if $p^2$ is even, then $p$ is even. Can you prove this?

The main proof continues with the deduction that if $p$ is even there must be another *integer, r*, that is half $p$.

$$p=2r\Rightarrow p^2=4r^2\Rightarrow 2q^2=4r^2$$
$$\Rightarrow q^2=2r^2\Rightarrow q^2 \text{ is even}\Rightarrow q \text{ is even}$$

We now have our contradiction as we assumed that $p/q$ was in its lowest terms so $p$ and $q$ cannot both be even. This proves the result, because we have a contradiction.

This theorem is a very strong statement of **impossibility**.

There are very few other areas of knowledge in which we can make similar statements. We may be virtually certain that we will never travel faster than the speed of light but it would be a brave physicist who would state with certainty that it is *impossible*. Other methods of proof include proof by induction which is mainly used to prove theorems involving sequences of statements.

Whilst on the subject of proof, it is worth noting that it is much easier to disprove a statement than to prove it. When we succeed in disproving a statement, we have succeeded in proving its negation or reverse. To disprove a statement, all we need is a single example of a case in which the theorem does not hold. Such a case is known as a **counter-example**.

The theorem 'all prime numbers are odd' is false. This can be established by noting that 2 is an even prime and, therefore, is the only counter-example we need to give. By this method we have proved the theorem that 'not every prime number is odd'.

This is another example of the way in which pure mathematicians think in a slightly different way from other disciplines. Zoo-keepers (and indeed the rest of us) may be happy with the statement that "all giraffes have long necks" and would not be very impressed with a pure mathematician who said that the statement was false because there was one giraffe (with a birth defect) who has a very short neck. This goes back to the slightly different standards of proof that are required in mathematics.

Counter-examples and proofs in mathematics may be difficult to find.

Consider the theorem that every odd positive integer is the sum of a prime number and twice the square of an integer. Examples of this theorem that do work are:

$5=3+2\times1^2$, $15=13+2\times1^2$, $35=17+2\times3^2$.

The theorem remains true for a very large number of cases and we do not arrive at a counter-example until 5777.

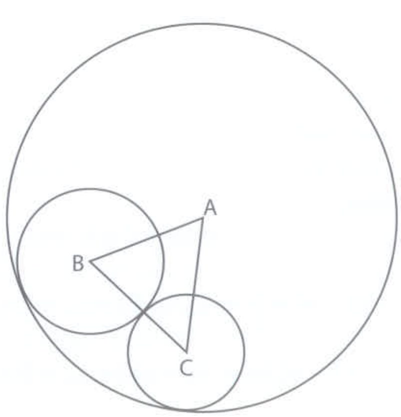
Another similar 'theorem' is known as the *Goldbach Conjecture*. Christian Goldbach (1690–1764) stated that every even number larger than 2 can be written as the sum of two primes. For example, $4=2+2$, $10=3+7$, $48=19+29$ etc. No-one has ever found a counter-example to this simple conjecture and yet no accepted proof has ever been produced, despite the fact that the conjecture is not exactly recent!

Finally, whilst considering proof, it would be a mistake to think that mathematics is a complete set of truths that has nothing which needs to be added. We have already seen that there are unproved theorems that we suspect to be true. It is also the case that new branches of mathematics are emerging with a fair degree of regularity. During this course you will study linear programming which was developed in the 1940s to help solve the problems associated with the distribution of limited resources. Recently, both pure and applied mathematics have been enriched by the development of 'Chaos Theory'. This has produced items of beauty such as the Mandelbrot set (see Chapter A7) and insights into the workings of nature. It seems, for example, that the results of Chaos Theory indicate that accurate long-term weather forecasts will never be possible (Mandelbrot).

The future shapes of these clouds is likely to be forever beyond the powers of mathematics.

### Exercise A.8.1

1. Prove that if $a \in \mathbb{Z}$ and $a^2 - 2a + 7$ is even the $a$ is odd.
2. Prove that $\sqrt{6}$ is irrational.
3. Prove that there are no integers $a$ and $b$ for which $21a + 30b = 1$.
4. If $a$ and $b$ are positive real numbers, then $a + b \leq 2ab$.
5. Prove that $\sqrt[3]{2}$ is irrational.
6. Prove that $a \in \mathbb{Z}$ is odd $\Leftrightarrow a^2$ is odd.
7. Prove that there is no largest even integer.
8. Prove that there do not exist integers $m$ and $n$ such that $14m + 21n = 100$.
9. Prove that triangle ABC can have no more than one right angle.
10. Prove that if $a$ is a rational number and $b$ is an irrational number, then $a + b$ is an irrational number.
11. Prove that there are no positive integer solutions to the equation $x^2 - y^2 = 10$.
12. Prove that there is no smallest positive rational number.
13. Prove that no odd integer can be expressed as the sum of three even integers.
14. Prove that a regular polyhedron cannot have hexagonal faces.
15. A, B & C are centres of the three circles. Prove that, irrespective of the sizes of the two small circles, the perimeter of $\Delta$ABC is constant.




# Mathematical Induction

Induction is an indirect method of proof which is used in cases where a direct method is either not possible or not convenient. It involves the derivation of a general rule from one or more particular cases, i.e. the general rule is *induced*. This is the opposite to deduction, where you use the general rule to provide detail about a particular case. For example, we know that 60 is divisible by 1, 2, 3, 4, 5 and 6, but does it follow that 60 is divisible by all positive integers?

**Example A.8.1**

Consider the pattern:

$$\begin{aligned} 1 &= 1^2 \\ 1+3 &= 2^2 \\ 1+3+5 &= 3^2 \\ 1+3+5+7 &= 4^2 \end{aligned}$$

The pattern shows that the sum of the first two positive odd integers is a perfect square, the sum of the first three positive odd integers is a perfect square and the sum of the first four positive odd integers is a perfect square. Can we then say, based on the first few lines of this pattern, that:

$1 + 3 + 5 + 7 + \ldots + (2n - 1) = n^2$

This can be checked, as the positive odd integers form an arithmetic progression (see Core A.2) with $a = 1$ and $d = 2$. The sum of the first $n$ terms is given by:

$$S_n = \frac{n}{2}(a + l) \text{ where } a \text{ is the first term, } l \text{ is the last term.}$$

$$= \frac{n}{2}(1 + 2n - 1)$$

$$= n^2$$

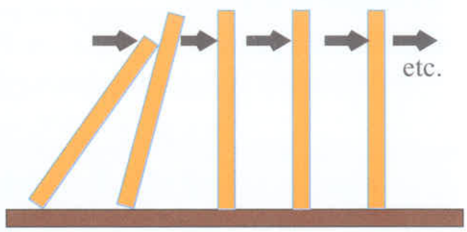
In this case the general result was easy to guess, but remember that a guess is not a proof. Thankfully in this example we had a method (sum of an AP) to verify our guess. This will not always be the case.

Consider the expression $4n^3 - 18n^2 + 32n - 15$ for values of $n$ from 1 to 4 as shown in the table below:

| $n$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $4n^3 - 18n^2 + 32n - 15$ | 3 | 9 | 27 | 81 |

The expression appears to produce the successive powers of 3, and so we could assume, based on the results in this table, that $4n^3 - 18n^2 + 32n - 15 = 3^n$.

Can we then say that this will always be the case, and if so, what would you predict the value of the expression to be when $n = 5$? Check to see if your prediction is correct.




Many formulae which we may guess or develop from simple cases can be proved using the **principle of mathematical induction**.

This method of proof relies upon a similar principle to that of 'domino stacking'. In the process of domino stacking, one domino is first pushed over, thus causing a series of dominoes to fall. Before each successive domino will fall, the preceding domino must fall.

With induction, for each expression to be true, the expression before it must also be true. The process can be summarized into four steps:

**Step 1:** the first expression must be true (the first domino falls)

**Step 2:** assuming that a general expression is true (*assume that some domino in the series falls*)

**Step 3:** prove that the next expression is true (*prove that the next domino in the series falls*)

**Step 4:** if all of these events happen then we know by induction that all of the expressions are true and thus the original formula is true (*all the dominoes will fall*).

**Example A.8.2**

Prove by induction that the sum of the first $n$ odd numbers is $n^2$.

That is, $1 + 3 + 5 + 7 + \ldots + (2n - 1) = n^2$ for all $n \geq 1$.

First we need to state what our proposition is. We do this as follows:

Let $P(n)$ be the proposition that $1 + 3 + 5 + 7 + \ldots + (2n - 1) = n^2$ for all $n \neq 1$.

Next we proceed with our four steps:

**Step 1:** test for $n = 1$

LHS = 1 AND RHS = $1^2 = 1$, $\therefore$ LHS = RHS

$\therefore$ the proposition $P(n)$ is true for $n = 1$ (the first domino falls!)

**Step 2:** assume that $P(n)$ is true for $n = k$ (a general domino falls)

i.e. $1 + 3 + 5 + 7 + \ldots + (2k - 1) = k^2$

**Step 3:** test the proposition for $n = k + 1$ (prove that the next domino falls)

i.e. we wish to prove that $1 + 3 + 5 + \ldots$
$\ldots + (2k - 1) + \{2(k + 1) - 1\} = (k + 1)^2$

Now, LHS $= 1 + 3 + 5 + \ldots + (2k - 1) + (2k + 1)$

$= k^2 + (2k + 1)$ (as $1 + 3 + 5 + \ldots + (2k - 1) = k^2$ (from Step 2)

$= (k + 1)^2$

= RHS $\therefore$ $P(n)$ is true for $n = k + 1$

**Step 4:** Thus, if the proposition is true for $n = k$ (Step 2), then it is true for $n = k + 1$. As it is true for $n = 1$, then it must be true for $n = 1 + 1$ ($n = 2$). As it is true for $n = 2$ then it must hold for $n = 2 + 1$ ($n = 3$) and so on for all positive integers $n$.

An alternative way of looking at mathematical induction is to think of the problem as a series of assertions. If the first assertion is true, and then each assertion which is true is followed by a true assertion, then all of the assertions in the sequence are true.

**Example A.8.3**

Prove that the formula

$$1 + 2 + 3 + \dots + n = \frac{1}{2}n(n+1)$$

is true for all positive integral $n$.

**Step 1:** The formula is actually a series of assertions:

$$n = 1: \quad 1 = \frac{1}{2} \times 1 \times 2$$

$$n = 2: \quad 1 + 2 = \frac{1}{2} \times 2 \times 3$$

$$n = 3: \quad 1 + 2 + 3 = \frac{1}{2} \times 3 \times 4 \text{ etc.}$$

The first assertion is obviously true so we now need to prove that the assertion following each true assertion is itself true.

**Step 2:** Suppose the $k^{\text{th}}$ assertion is true,

i.e. $1 + 2 + 3 + \dots + k = \frac{k(k+1)}{2}$

**Step 3:** Now add the $(k+1)^{\text{th}}$ term i.e. $(k+1)$ to both sides of this equation, obtaining:

$$1 + 2 + 3 + \dots + k + (k+1) = \frac{k(k+1)}{2} + (k+1)$$

$$= \frac{k(k+1) + 2(k+1)}{2}$$

$$= \frac{(k+1)(k+2)}{2}$$

**Step 4:** But this is equivalent to the $(k+1)^{\text{th}}$ assertion, which is true if the $k^{\text{th}}$ assertion is true. We have thus shown that the assertion following each true assertion is also true, and thus by mathematical induction the formula given is true for all $n$.

## Exercise A.8.2

Prove by induction that for all $n$:

a $\quad 1 + 4 + 7 + \dots + (3n-2) = \frac{1}{2}n(3n-1)$

b $\quad 1 + 5 + 9 + \dots + (4n-3) = n(2n-1)$

c $\quad 2 + 4 + 6 + \dots + 2n = n(n+1)$

d $\quad 5 + 10 + 15 + \dots + 5n = \frac{5}{2}n(n+1)$

e $\quad 6 + 12 + 18 + \dots + 6n = 3n(n+1)$

f $\quad 1 + 2 + 4 + 8 + \dots + 2^{n-1} = 2n - 1$

g $\quad 1 + r + r^2 + \dots + r^{n-1} = \frac{1-r^n}{1-r}$

h $\quad 1^2 + 3^2 + 5^2 + \dots + (2n-1)^2 = \frac{1}{3}n(4n^2 - 1)$

i $\quad 1^2 - 2^2 + 3^2 - \dots + (-1)^{n-1}n^2 = \frac{1}{2}(-1)^n - 1n(n+1)$

In this section we consider some propositions involving divisibility and inequalities.

**Example A.8.4**

Prove by induction that $9^n - 1$ is divisible by 8 for all $n \geq 1$ where $n$ is an integer.

Let $P(n)$ be the proposition that $9^n - 1|8$ (i.e. $9^n - 1$ is divisible by 8) for all $n \geq 1$.

**Step 1:** The proposition is true when $n = 1$ since $9^1 - 1 = 8$ which is divisible by 8

**Step 2:** Assume $P(n)$ holds true for $n = k$, i.e. assume that $9^k - 1 = 8m$ where $m$ is an integer.

**Step 3:** Prove $P(n)$ is true for $n = k + 1$. i.e. prove that $9^{k+1} - 1$ is divisible by 8.

Now $9^{k+1} - 1 = 9(9^k) - 1$

$= 9(8m + 1) - 1$ (as $9^k = 8m + 1$ (from Step 2))

$= 72m + 8$

$= 8(9m + 1)$ which is divisible by 8

Therefore, $P(n)$ is true for $n = k + 1$.

**Step 4:** That is, if the proposition holds for $n = k$, it also holds for $n = k + 1$. As it is true for $n = 1$ it is then true for $n = 2$, and so on, and thus the proposition is true for all $n \geq 1$.

**Example A.8.5**

Prove by induction that $2^n > n$ for all $n \geq 1$.

Let $P(n)$ be the proposition that $2^n > n$ for all $n \geq 1$.

**Step 1:** $P(n)$ is true when $n = 1$ since

$$\text{L.H.S} = 2^1 = 2 > 1 = \text{R.H.S}$$

**Step 2:** Assume that $P(n)$ holds for $n = k$; i.e. that $2^k > k$

**Step 3:** Prove that $P(n)$ is true for $n = k + 1$

i.e. show that $2^{k+1} > k + 1$.

From Step 2 above, $2^k > k$

$2 \times 2^k > 2k$ (multiplying both sides by 2)

But, $2 \times 2^k = 2^{k+1}$ $\quad \therefore 2^{k+1} > 2k$

Now, $k \geq 1$ so $2k = k + k \geq k + 1$ and hence $2^{k+1} > k + 1$

i.e. $P(n)$ holds for $n = k + 1$ if it holds for $n = k$.

**Step 4:** Thus as $P(n)$ holds for $n = 1$, it holds for $n = 1 + 1$ and so on for all values of $n \geq 1$.

## Exercise A.8.3

By induction, prove that:

a $\quad 9^{n+2} - 4^n$ is divisible by 5 for all $n \geq 1$

b $\quad n^3 - n$ is divisible by 3 for all $n > 1$

c $\quad n^3 + 2n$ is a multiple of 3 for all $n \geq 1$

d $\quad 7^n + 2$ is divisible by 3 for all $n \geq 1$

e $\quad 9^{n+1} - 8n - 9$ is divisible by 64 for all $n \geq 1$

f $\quad 2^n \geq 1 + n$ for all $n \geq 1$

Extra questions


We now consider more difficult propositions.

**Example A.8.6**

Prove by induction that $n^3 + 5n$ is divisible by 6 for all $n \geq 1$.

Let $P(n)$ be the proposition that $n^3 + 5n$ is divisible by 6 for all $n \geq 1$.

**Step 1:** Test for $n = 1$

$1^3 + 5 \times 1 = 6$ which is divisible by 6 and so the proposition is true for $n = 1$

**Step 2:** Let $P(n)$ be true for $n = k$,

i.e. $\dfrac{k^3 + 5k}{6} = m \Leftrightarrow k^3 + 5k = 6m$, $m$ is an integer.

**Step 3:** Test for $n = k + 1$

$$\begin{aligned}(k+1)^3 + 5(k+1) &= k^3 + 3k^2 + 3k + 1 + 5k + 5 \\ &= (k^3 + 5k) + 3k^2 + 3k + 6 \\ &= 6m + 3k^2 + 3k + 6 \text{ (from Step 2)} \\ &= 6m + 6 + 3k(k+1)\end{aligned}$$

Now $k(k + 1)$ is an even number and thus it has a factor of 2 (the product of two consecutive integers is even). Thus the product $3k(k + 1)$ can be written as $3 \times 2 \times q$ where $q$ is the quotient of $k(k + 1)$ and 2.

$$\begin{aligned}\therefore \text{LHS} &= 6m + 6 + 6q \\ &= 6(m + 1 + q) \text{ which is divisible by 6.}\end{aligned}$$

**Step 4:** Thus, if the proposition is true for $n = k$ then it is true for $n = k + 1$ as proved. As it is true for $n = 1$, then it must be true for $n = 1 + 1$ ($n = 2$). As it is true for $n = 2$ then it must hold for $n = 2 + 1$ ($n = 3$) and so on for all positive integers $n$.

That is, by the principle of mathematical induction $P(n)$, is true.

**Example A.8.7**

Prove that:

$$\sum_{r=1}^{n} (4r-6) = 2n(n-2) \text{ for all } n \in \mathbb{Z}^+$$

Let $P(n)$ be the proposition that

$$\sum_{r=1}^{n} (4r-6) = 2n(n-2) \text{ for all } n \in \mathbb{Z}^+.$$

However, when dealing with sigma notation it can be helpful to write the first few terms of the sequence:

$$\sum_{r=1}^{n} (4r-6) = -2+2+6+10+\ldots+(4n-6) = 2n(n-2)$$

**Step 1:** $P(n)$ is true for $n = 1$ since
L.H.S $= 4 \times 1 - 6 = 2 \times (1-2) = -2 =$ R.H.S.

**Step 2:** Assume that $P(n)$ is true for $n = k$,

i.e. $\sum_{r=1}^{k} (4r-6) = 2k(k-2)$.

**Step 3:** Test $P(n)$ for $n = k + 1$:

Adding the $(k+1)^{\text{th}}$ term, $[4(k+1)-6]$ to both sides gives

$$\sum_{r=1}^{k} (4r-6) + [4(k+1)-6] = 2k(k-2) + [4(k+1)-6]$$

(from Step 2)

$$= 2k^2 - 4k + 4k - 2$$

$$= 2(k^2 - 1)$$

$$= 2(k+1)(k-1)$$

$$= 2(k+1)[(k+1)-2]$$

which is the $(k+1)^{\text{th}}$ assertion.

That is, $P(n)$ is true for $n = k + 1$.

**Step 4:** Thus, if the proposition is true for $n = k$, then it is true for $n = k + 1$. As it is true for $n = 1$, then it must be true for $n = 1 + 1$ ($n = 2$). As it is true for $n = 2$ then it must hold for $n = 2 + 1$ ($n = 3$) and so on for all positive integers $n$.

That is, by the principle of mathematical induction, $P(n)$ is true.

## Exercise A.8.4

Prove the following using the principle of mathematical induction for all $n \in \mathbb{Z}^+$.

a $\displaystyle\sum_{r=1}^{n} \frac{1}{(2r-1)(2r+1)} = \frac{n}{2n+1}$

b $\displaystyle\sum_{r=1}^{n} (2r-1)^3 = n^2(2n^2-1)$

c $\displaystyle\sum_{r=1}^{n} 5^r = \frac{1}{4}(5^{n+1}-5)$

d $2 + 2^2 + 2^3 + \ldots + 2^n = 2(2^n - 1)$

e $1 + 3 + 9 + \ldots + 3^{n-1} = \frac{1}{2}(3^n - 1)$

f $1 + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^{n-1}} = 2 - 2^{1-n}$

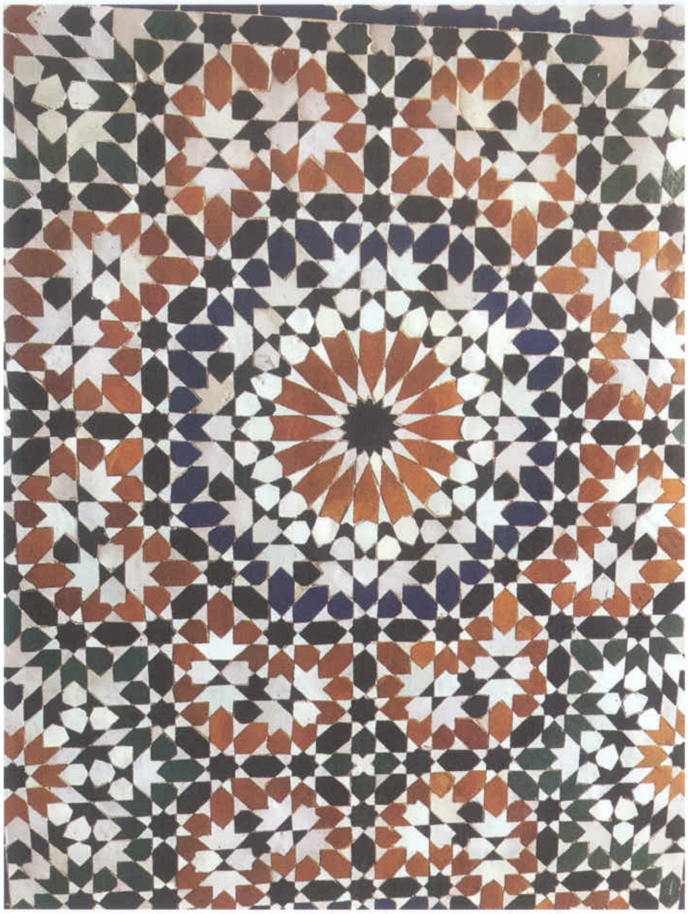
g $1.1 + 2.3 + 3.5 + \ldots + n(2n-1) = \frac{1}{6}n(n+1)(4n-1)$

h $1.1 + 3.2 + 5.4 + \ldots + (2n-1)2^{n-1} = 3 + 2^n(2n-3)$

Extra questions

Mathematics can be considered to be the study of patterns. A useful ability in maths can be forming a rule to describe a pattern. Of course any rule that we develop must be true in all relevant cases and mathematical induction provides one method of proof.

Patterns are not confined to tessellations!



Here are two final examples:

**Example A.8.8**

Find the sum to $n$ terms of the series $\frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^n}$

Consider $f(n) = \frac{a}{2^n}$ then: $f(n+1) = \frac{a}{2^{n+1}}$.

Now $u_n = f(n+1) - f(n)$ , i.e. $\frac{1}{2^n} = \frac{a}{2^{n+1}} - \frac{a}{2^n}$.

Equating and solving for $a$ gives $a = -2$

$\therefore f(n+1) = \frac{-2}{2^{n+1}} = -\frac{1}{2^n}$, $f(1) = -1$ .

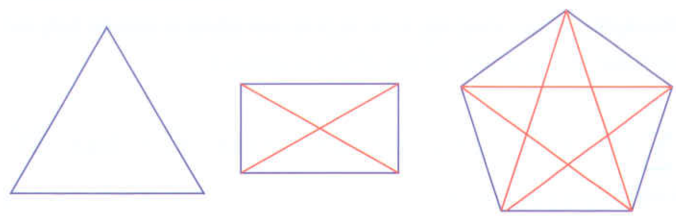
Therefore,

$$\frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^n} = f(n+1) - f(1) = -\frac{1}{2^n} + 1 = 1 - \frac{1}{2^n} .$$

This result can be proved by induction.

**Example A.8.9**

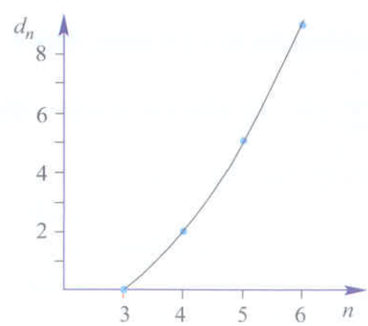
Find the number of diagonals that can be drawn in an $n$-sided convex polygon.



Let $d_n$ represent the number of diagonals in an $n$-sided polygon. The value for $d_n$ is shown in the table for values of $n$ up to $n = 6$ (*construct the next few diagrams in the pattern to verify and extend this table*).

| $n$ | 3 | 4 | 5 | 6 |
|---|---|---|---|---|
| $d_n$ | 0 | 2 | 5 | 9 |

Plotting the points related to the variables $n$ and $d_n$ (*above*) suggests that the relationship between them *could* be quadratic, and so we might assume that




$$d_n = an^2 + bn + c$$

Substituting the first 3 values for $n$ gives:

$n = 3 \quad \Rightarrow 0 = 9a + 3b + c$

$n = 4 \quad \Rightarrow 2 = 16a + 4b + c$

$n = 5 \quad \Rightarrow 5 = 25a + 5b + c$

Solving these three equations for $a$, $b$ and $c$ gives

$a = \frac{1}{2}$, $b = -\frac{3}{2}$, and $c = 0$ and thus $d_n = \frac{1}{2}n^2 - \frac{3}{2}n = \frac{n(n-3)}{2}$.

When $n = 6$, $d_6 = \frac{6(6-3)}{2} = 9$, which corresponds to the tabulated value for $n = 6$ above.

So far we have formed a **conjecture** that the number of diagonals in an $n$-sided convex polygon is given by $d_n = \frac{n(n-3)}{2}$. This formula remains a conjecture until we prove that it is true for values of $n \geq 3$.

**Proof:**

Let $P(n)$ be the proposition that the number of diagonals that can be drawn in an $n$-sided convex polygon is given by $d_n = \frac{n(n-3)}{2}$ for $n \geq 3$.

**Step 1:** $P(n)$ is true for $n = 3$ as $d_3 = \frac{3(3-3)}{2} = 0$ which is the number of diagonals in a 3-sided polygon.

**Step 2:** Assume that $P(n)$ is true for a $k$-sided polygon i.e. that $d_k = \frac{k(k-3)}{2}$. We consider the effect that adding an extra side will have on the result.

**Step 3:** Looking at the tabulated values for $n$ and $d_n$ you should see that adding an extra side to an $n$-sided polygon produces an extra $(n-1)$ diagonals, and so we can say that

$$
\begin{aligned}
d_{k+1} &= d_k + \text{the extra diagonals added by the extra side} \\
&= d_k + (k-1) \\
&= \frac{k(k-3)}{2} + (k-1) \\
&= \frac{k(k-3) + 2(k-1)}{2} \\
&= \frac{(k+1)(k-2)}{2} \\
&= \frac{(k+1)[(k+1)-3]}{2}
\end{aligned}
$$

**Step 4:** Which is the $(k+1)^{th}$ assertion.

Thus, if the proposition is true for $n = k$, then it is true for $n = k + 1$. As it is true for $n = 3$, then it must be true for $n = 3 + 1$ ($n = 4$). As it is true for $n = 4$ then it must hold for $n = 4 + 1$ ($n = 5$) and so on for all integers $n \geq 3$.

By the principle of mathematical induction, $P(n)$ is true.

## Exercise A.8.5

Find the sum to $n$ terms of the sequences below and then prove your results true.

**a** $2 + 5 + 10 + 17 + \ldots + (n^2 + 1)$

**b** $1 + 8 + 27 + 64 + \ldots + n^3$

**c** $\frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \ldots + \frac{1}{5^n}$

**d** $1^3 + 3^3 + 5^3 + \ldots + (2n-1)^3$

**e** $1.3 + 2.4 + 3.5 + \ldots + n(n+2)$

**f** $\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \ldots + \frac{1}{(2n-1)(2n+1)}$

For questions 2 to 6, find the required general result and then prove your answer using mathematical induction.

2. 1, 3, 6, 10, 15, … are called triangular numbers.

Denoting the $n$th triangular number as $t_n$, find a formula for $t_n$.

3. Find the size of each angle in a regular $n$-sided polygon.

4. Find the maximum number of pieces that can be formed making $n$ straight cuts across a circular pizza (*pieces don't have to be of equal size*).

5. Find the number of squares of all sizes on an $n \times n$ chess board.

6. Prove that a three digit number is divisible by 3 if the sum of its digits is divisible by 3.

Extra questions

Answers will vary.

# Postscript

## Counting rabbits

Mathematicians are searchers after pattern. This reflects an innate human proclivity for looking for connections even when none exist. There is nothing the tabloid press loves more than a peasant who finds the face of the US president when they slice open a watermelon. However, most of these 'connections' have no actual meaning.

Can the same be said of mathematical connections?

Here are the first few rows of what is variously called the "Chinese triangle" or "Pascal's triangle":

```
                  1
               1     1
            1     2     1
         1     3     3     1
      1     4     6     4     1
   1     5    10    10     5     1
1     6    15    20    15     6     1
```

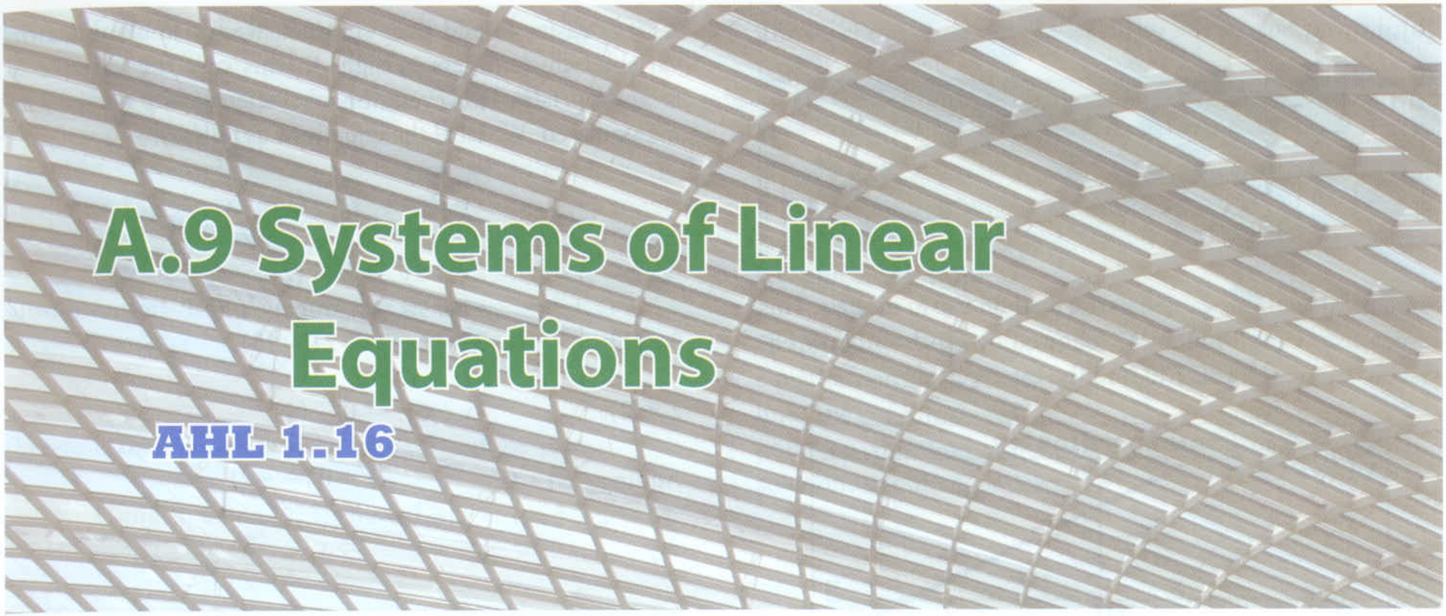
Now displace each row to the right to produce the echelon form shown below and sum the columns (only the first seven columns are complete).

```
1
  1 1
    1 2 1
      1 3 3 1
        1 4 6 4 1
          1 5 10 10 5 1
            1 6 15 20 15 6 1
              1 .....
_______________________________
1 1 2 3 5 8 13 ...
```

It looks like we have the Fibonacci sequence (and the rabbits) again. How can you be certain that this is not just chance and that the pattern continues forever.

What distinguishes the true mathematician from the presidential watermelloners is that a mathematician will demand a proof. Can you supply it?

And once you have a proof, does this imply that the polynomial coefficients are really connected to the mating habits of rabbits?



# A.9 Systems of Linear Equations

AHL 1.16

## Simultaneous linear equations in two unknowns

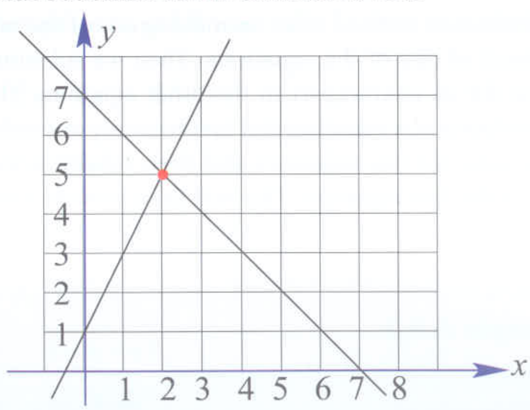
Pairs of simultaneous equations in two unknowns may be solved in two ways, either algebraically or graphically. To solve means to find where the two straight lines intersect once they have been sketched. So, we are looking for the point of intersection.

### Method 1: Graphical

**Example A.9.1**

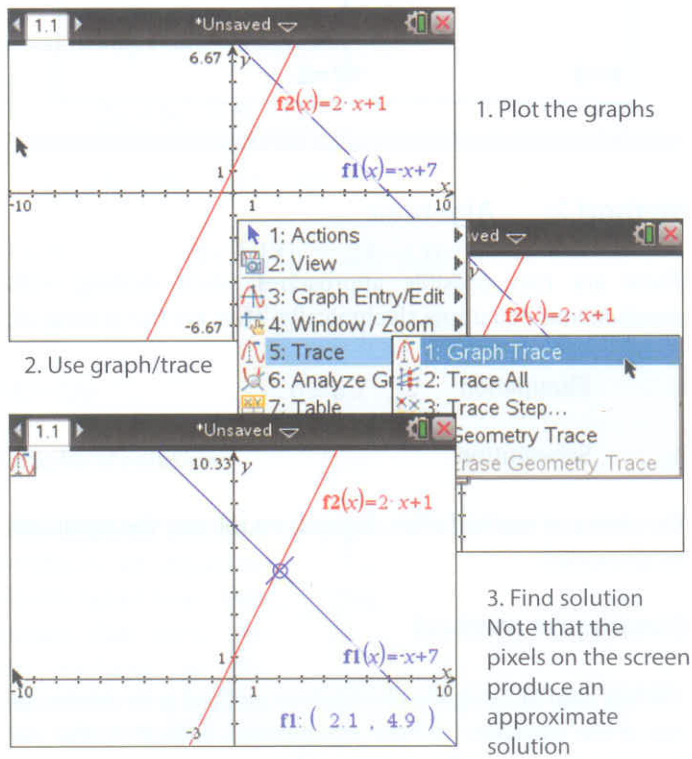
Solve the system of linear equations $y = -x + 7$ and $y = 2x + 1$.

We sketch both lines on the same set of axes:

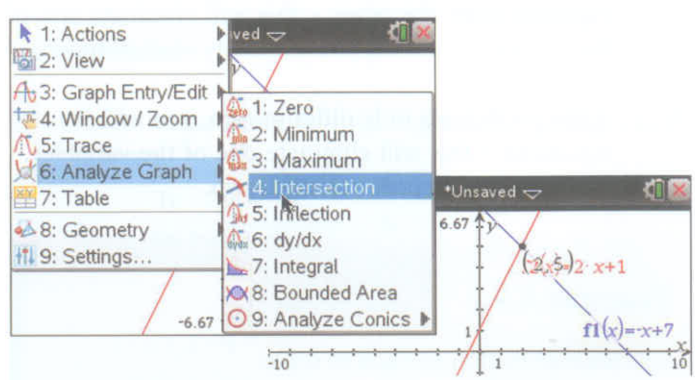



Reading off the grid we can see that the straight lines meet at the point with coordinates (2, 5). So, the solution to the given system of equations is $x = 2$ and $y = 5$.

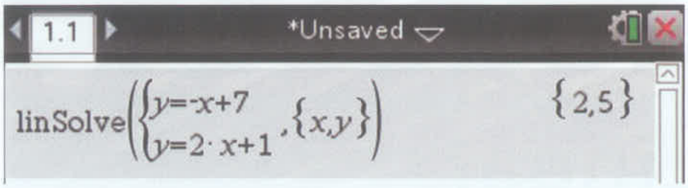
There are a number of ways that the graphics calculator can be used.



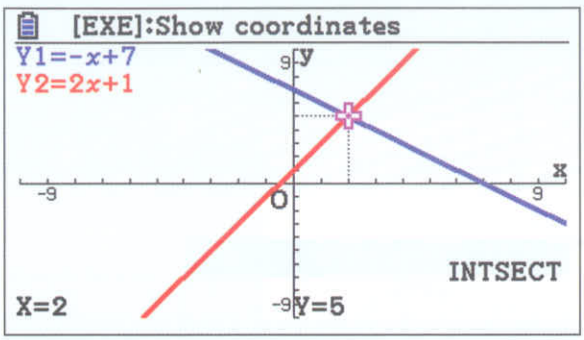

A more satisfactory way is to use the calculator to find the intersection.



Thirdly, you can use the solve facility:




Similar calculations can be performed on Casio models. This screen uses Graph mode (5) followed by F6-draw, Shift F5-G-Solv and F5-INTSCT to find the intersection.




## Method 2: Algebraic

There are two possible approaches when dealing with simultaneous equations algebraically. They are the process of:

**1.** Elimination

**2.** Substitution

The choice of method often depends on the way the equations are presented.

### Elimination method

The **key step** in using the elimination method is to obtain, for one of the variables (in both equations), coefficients that are the same (or only differ in sign). Then:

1. if the coefficients are the same, you subtract one equation from the other – this will eliminate one of the variables – leaving you with only one unknown.

2. if the coefficients only differ in sign, you add the two equations – this will eliminate one of the variables – leaving you with only one unknown.

**Example A.9.2**

Use the elimination method to solve: $\begin{array}{l} x-2y=-7 \\ 2x+3y=0 \end{array}$

As it is easier to add than subtract, we try to eliminate the variable which differs in sign. In this case the variable '$y$' is appropriate. However, the coefficients still need to be manipulated. We label the equations as follows:

$$x-2y = -7 \quad -(1)$$

$$2x+3y = 0 \quad -(2)$$

$3\times(1)$: $\quad 3x-6y = -21 \quad -(3)$

$2\times(2)$: $\quad 4x+6y = 0 \quad -(4)$

Adding $(3)+(4)$: $\quad 7x+0 = -21$

$$\Leftrightarrow x = -3$$

Substituting into (1) we can now obtain the $y$-value:

$$-3-2y = -7 \Leftrightarrow -2y = -4 \Leftrightarrow y = 2.$$

Therefore, the solution is $x = -3$, $y = 2$.

Once you have found the solution, always check with one of the original equations.

Using equation (2) we have: L.H.S $= 2\times-3+3\times2 = 0 =$ R.H.S.

Note that we could also have multiplied equation (1) by 2 and then subtracted the result from equation (2). Either way, we have the same answer.

### Substitution method

The substitution method relies on making one of the variables the subject of one of the equations. Then we substitute this equation for its counterpart in the other equation. This will then produce a new equation that involves only one unknown. We can solve for this unknown and then substitute its value back into the first equation. This will then provide a solution pair.

**Example A.9.3**

Use the substitution method to solve: $\begin{array}{l} 5x-y=4 \\ x+3y=4 \end{array}$

Label the equations as follows: $\quad 5x-y = 4 \quad -(1)$

$$x+3y = 4 \quad -(2)$$

From equation (1) we have that $y = 5x - 4$ - (3)

Substituting (3) into (2) we have: $x + 3(5x - 4) = 4$

$\Leftrightarrow 16x - 12 = 4$

$\Leftrightarrow 16x = 16$

$\Leftrightarrow x = 1$

Substituting $x = 1$ into equation (3) we have:

$$y = 5 \times 1 - 4 = 1$$

Therefore, the solution is given by $x = 1$ and $y = 1$.

Check: Using equation (2) we have: L.H.S $= 1 + 3 \times 1 = 4$ = R.H.S.

Not all simultaneous equations have unique solutions. Some pairs of equations have no solutions while others have infinite solution sets. You will need to be able to recognise the 'problem' in the processes of both algebraic and graphical solutions when dealing with such equations.

The following examples illustrate these possibilities.

**Example A.9.4**

Solve: a $\begin{matrix} 2x+6y=8 \\ 3x+9y=12 \end{matrix}$ b $\begin{matrix} 2x+6y=8 \\ 3x+9y=15 \end{matrix}$

a **Algebraic solution:**

Label the equations as follows:

$2x + 6y = 8$ - (1)

$3x + 9y = 12$ - (2)

$3 \times (1)$: $6x + 18y = 24$ - (3)

$2 \times (2)$: $6x + 18y = 24$ - (4)

In this case, we have the same equation. That is, the straight lines are **coincident**.

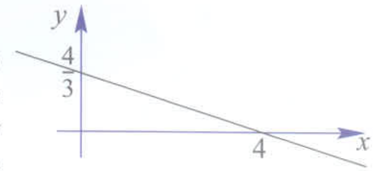
If we were to 'blindly' continue with the solution process, we would have:

$3 \times (1) - 2 \times (2)$: $0 = 0$ ✓

The algebraic method produces an equation that is always true, i.e. zero will always equal zero. This means that any pair of numbers that satisfy either equation will satisfy both and are, therefore, solutions to the problem. Examples of solutions are: $x = 4, y = 0, x = 1, y = 1, x = 7, y = -1$. In this case we say that there is an **infinite** number of solutions.

**Graphical solution:**

Graphically, the two equations produce the same line. The coordinates of any point on this line will be solutions to both equations.




b **Algebraic solution:**

Label the equations as follows:

$2x + 6y = 8$ - (1)

$3x + 9y = 15$ - (2)

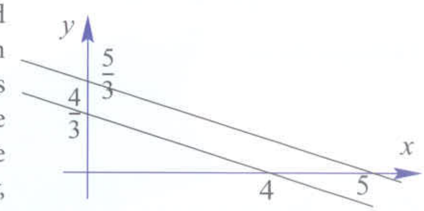
$3 \times (1)$: $6x + 18y = 24$ - (3)

$2 \times (2)$: $6x + 18y = 30$ - (4)

(4) – (3): $0 = 6$ ✗

**Graphical solution:**

The algebraic method produces an equation that is never true. This means that there are no solutions to the equations. Graphically, the two lines are parallel and produce no points of intersection.




## Exercise A.9.1

1. Solve these simultaneous equations, giving exact answers.

a $\begin{matrix} 3x - 2y = -1 \\ 5x + 2y = 9 \end{matrix}$ b $\begin{matrix} 3x + 5y = 34 \\ 3x + 7y = 44 \end{matrix}$

c $\begin{matrix} 2x + 4y = 6 \\ 4x - 3y = -10 \end{matrix}$ d $\begin{matrix} 3x + 2y = 2 \\ 2x - 6y = -6 \end{matrix}$

e $\begin{aligned} 5x+4y &= -22 \\ 3x-y &= -3 \end{aligned}$ f $\begin{aligned} 5x-9y &= -34 \\ 2x+3y &= -7 \end{aligned}$

2. Solve these simultaneous equations, giving fractional answers where appropriate.

a $\begin{aligned} 3x-y &= 2 \\ 5x+2y &= 9 \end{aligned}$ b $\begin{aligned} 4x+2y &= 3 \\ x-3y &= 0 \end{aligned}$

c $\begin{aligned} -3x+y &= 0 \\ 2x-4y &= 0 \end{aligned}$ d $\begin{aligned} \frac{x}{2}-3y &= 4 \\ 4x+\frac{3y}{2} &= -1 \end{aligned}$

e $\begin{aligned} 5x+\frac{2y}{3} &= -4 \\ 4x+y &= 2 \end{aligned}$ f $\begin{aligned} \frac{3x}{5}-4y &= \frac{1}{2} \\ x-2y &= \frac{1}{3} \end{aligned}$

3. Find the values of $m$ such that these equations have no solutions.

a $\begin{aligned} 3x-my &= 4 \\ x+y &= 12 \end{aligned}$ b $\begin{aligned} 5x+y &= 12 \\ mx-y &= -2 \end{aligned}$

c $\begin{aligned} 4x-2y &= 12 \\ 3x+my &= 2 \end{aligned}$

4. Find the values of $m$ and $a$ such that these equations have infinite solution sets.

a $\begin{aligned} 4x+my &= a \\ 2x+y &= 4 \end{aligned}$ b $\begin{aligned} 5x+2y &= 12 \\ mx+4y &= a \end{aligned}$

c $\begin{aligned} 3x+my &= a \\ 2x-4y &= 6 \end{aligned}$

Extra questions

## Simultaneous linear equations in three unknowns

So far we have looked at linear equations in two unknowns. However, this can be extended to linear equations in three unknowns. Equations such as these, involving the variables $x$, $y$ and $z$ take on the general form $ax+by+cz=k$ where $a$, $b$, $c$ and $k$ are real constants.

Just as for the case with two unknowns, where we required two equations to (hopefully) obtain a unique solution to the system of simultaneous equations, when dealing with three unknowns we will require a minimum of three equations to (hopefully) obtain a unique solution.

The solution process for a system of linear equations in three unknowns will require, primarily, the use of the elimination method. The method usually involves the reduction of a system of three equations in three unknowns to one of two equations in two unknowns. This will then enable the use of the methods already discussed to solve the 'reduced' system. Once two of the unknowns have been determined from this 'reduced' system, we substitute back into one of the original three equations to solve for the third unknown.

**Example A.9.5**

Solve the simultaneous equations: $\begin{aligned} x+3y-z&=13 \\ 3x+y-z&=11 \\ x+y-3z&=11 \end{aligned}$

We label the equations as follows:

$$x+3y-z = 13 \quad -(1)$$

$$3x+y-z = 11 \quad -(2)$$

$$x+y-3z = 11 \quad -(3)$$

Reduce the system to one involving two equations and two unknowns.

We first eliminate the variable $z$:

(2) – (1): $\quad 2x-2y = -2 \quad -(4)$

3 × (2) – (3): $\quad 8x+2y = 22 \quad -(5)$

Solve the reduced system of equations.

(4) + (5): $\quad 10x = 20 \quad \Leftrightarrow x = 2$

Substitute into (4):
$2\times 2-2y = -2 \Leftrightarrow -2y = -6 \Leftrightarrow y = 3$.

Solve for the third unknown.

Substituting $x=2$ and $y=3$ into (1):

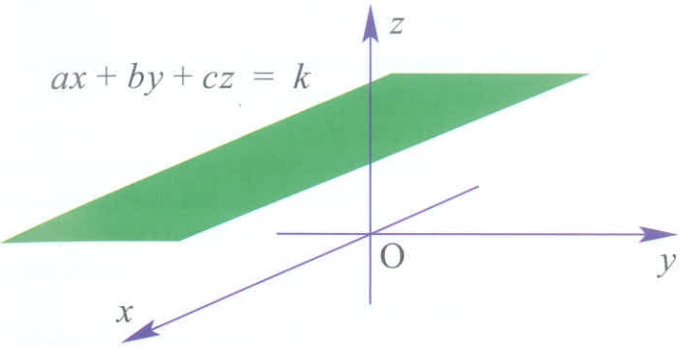
$2+3\times 3-z = 13 \Leftrightarrow z = -2$

Therefore the solution is given by $x=2$, $y=3$ and $z=-2$.

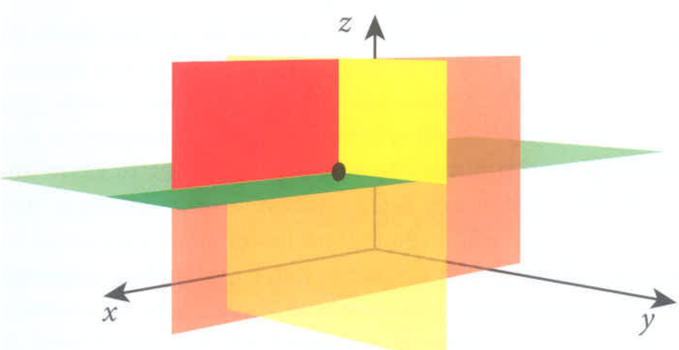
Check: Using equation (2):
L.H.S. = $2+3-3\times -2 = 11$ = R.H.S

We have already seen that linear equations in two unknowns are represented by straight lines on the Cartesian axes. The question then becomes, "What do linear equations in three unknowns look like?"

Equations of the form $ax + by + cz = k$ represent a plane in space. To draw such a plane we need to set up three mutually perpendicular axes that coincide at some origin O. This is commonly drawn with a horizontal $x$–$y$ plane and the $z$-axis in the vertical direction:

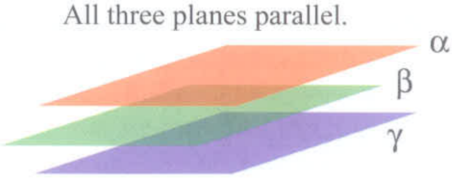



In Example A.9.5 we obtained a unique solution. This (2, 3, –2) means that the three planes must have intersected at a unique point. We can represent such a solution as shown in the diagram below:



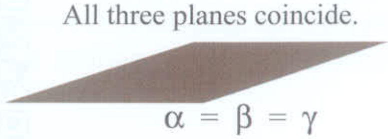

There are a number of possible combinations for how three planes in space can intersect (or not). Labelling the planes as α, β and γ the possible outcomes are shown below.

All three planes parallel.

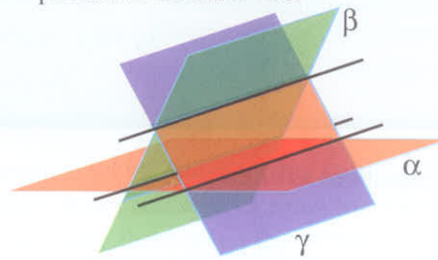



Two planes coincide.

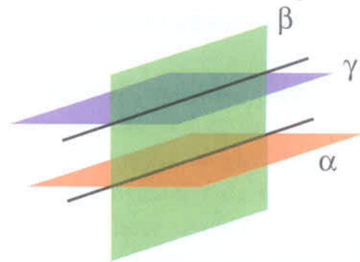

All three planes coincide.



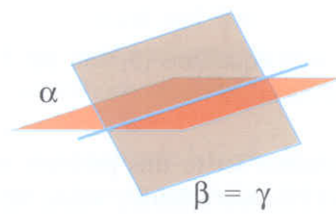

Any line of intersection is parallel to the other two.



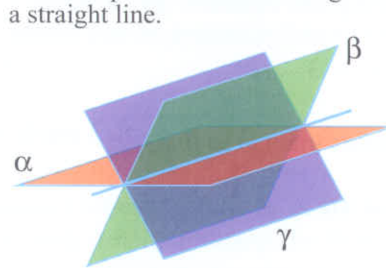

Two parallel non-coincident planes crossed by the third plane.



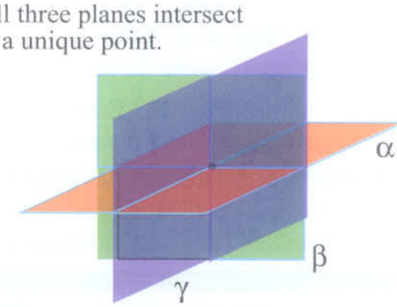

Two parallel coincident planes crossed by the third plane.




All three planes intersect along a straight line.




All three planes intersect at a unique point.



**Example A.9.6**

Solve the simultaneous equations:

$$x+2y=10$$
$$3x+2y-4z=18$$
$$y+z=3$$

We label the equations as follows;

$x+2y = 10$ – (1)

$3x+2y-4z = 18$ – (2)

$y+z = 3$ – (3)

We eliminate $x$ using equations (1) and (2):

(2) – 3 × (1): $-4y-4z = -12$

$\Leftrightarrow y+z = 3$ – (4)

We are now left with equations (3) and (4). However, these two equations are identical.

To obtain the solution set to this problem we introduce a **parameter,** we let $z$ be any arbitrary value, say $z = k$ where $k$ is some real number.

Then, substituting into equation (4), we have:

$y+k = 3 \Rightarrow y = 3-k$.

Next, we substitute into (1) so that

$x+2(3-k) = 10 \Rightarrow x = 4+2k$.

Therefore, the solution is given by

$x = 4+2k, y = 3-k, z = k$.

Notice the nature of the solution. Each of the variables is expressed as a linear function of $k$. This means that we have a situation where the three original planes meet along a straight line.

Supplementary example - matrices.

## Exercise A.9.2

Solve the simultaneous equations:

a
$$\begin{aligned} 6x+4y-z &= 3 \\ x+2y+4z &= -2 \\ 5x+4y &= 0 \end{aligned}$$

b
$$\begin{aligned} x+y+z &= 2 \\ 4x+y &= 4 \\ -x+3y+2z &= 8 \end{aligned}$$

c
$$\begin{aligned} 4x+9y+13z &= 3 \\ -x+3y+24z &= 17 \\ 2x+6y+14z &= 6 \end{aligned}$$

d
$$\begin{aligned} x-2y-3z &= 3 \\ x+y-2z &= 7 \\ 2x-3y-2z &= 0 \end{aligned}$$

e
$$\begin{aligned} x-y-z &= 2 \\ 3x+3y-7z &= 7 \\ x+2y-3z &= 3 \end{aligned}$$

f
$$\begin{aligned} x-2y &= -1 \\ -x-y+3z &= 1 \\ y-z &= 0 \end{aligned}$$

g
$$\begin{aligned} x+y+z &= 1 \\ x-y+z &= 3 \\ 4x+2y+z &= 6 \end{aligned}$$

h
$$\begin{aligned} -2x+y-2z &= 5 \\ x+4z &= 1 \\ x+y+10z &= 10 \end{aligned}$$

Answers

# Theory of Knowledge

## The Need for New Concepts and Notations

Throughout history, various notations and operations were introduced by mathematicians when they discovered their current set of notations was inadequate to address certain new mathematical concepts. In chapter A3 of the SL text, we studied the logarithm as an inverse operation to exponentiation.

The concept of exponentiation was used by Euclid as early as 300BC in ancient Greece. In other parts of the world, mathematicians continued to explore this concept and discovered new rules governing the proper use of exponents. However, it was not until the 17th century that the logarithm was first introduced by John Napier in a book titled *Mirifici Logarithmorum Canonis Descriptio.*

During the years when the idea of the logarithm was not formalized in the field of mathematics, were people not able to find the inverse of exponentiation? Taking this mathematical operation as an example, have you ever wondered what people used before a certain concept or notation was introduced?

In mathematics, as well as in other disciplines, before a concept was introduced and accepted to address a knowledge gap, does it mean that particular concept did not exist or was it irrelevant at that moment in time? What drives the discovery of a new concept in mathematics? Does intuition play a pivotal role in recognizing a gap in the current set of knowledge and notations before a mathematician can formalize and present a new concept, a new symbol, or a new notation? In other words, before a new mathematical concept is introduced did the empirical evidence or the rational thinking come first? Is it necessary before a new concept is formalized and accepted that it must have both empirical evidence and rational thinking?

With respect to the written notations in mathematics, have you ever wondered why certain symbols and notations are reused in different contexts and have very different meaning? For example, if you are presented with (3,5), does it suggest a coordinate pair on the Cartesian plane or does it suggest an open interval between 3 and 5 exclusively? Similarly, have you ever doubted your understanding of the difference between $f^{-1}(x)$ and $f(x)^{-1}$? These are just some examples to illustrate how a precise language like mathematics can also be ambiguous to a certain extent. If mathematics is a language, what grammatical rules are you following? Is this language evolving with time? Or is this language static and unchanging over time, thus limiting its ability to communicate newer concepts in mathematics?

## Assumptions and Conventions

By definition, an *assumption* is a claim for a concept, a thing, or a situation, that is accepted as true without evidence, justification, or proof. Conversely, by definition, *convention* is a way in which an action is usually taken or a way in which something is usually done.

In mathematics, when one attempts to provide an answer to a question, it is necessary to show the logical deductive reasoning to ensure there is no error in applying the algebraic rules. However, does it merely mean that assumptions and conventions are not to be used and considered when one studies mathematics? If one only provides an answer solely based on assumptions, does it mean that the answer is wrong?

It is understood by mathematicians that $x$ is $x^1$, when the exponent 1 is already assumed in the written notation of $x$. Similarly, the expression $\log x$ is assumed to be written in base 10 (i.e. $\log_{10}x$) or the radical term $\sqrt{x}$ is already understood to be the same as $\sqrt[2]{x}$ . Do these assumptions consequently affect the validity of the answer? Likewise, if these assumptions are generally accepted as a convention in written mathematics, then who decides which conventions are to be adopted or rejected? How do cultural and historical factors influence these assumptions and conventions? Are these written mathematical conventions infallible?

If mathematics is constructed from deductions, in which one must assume certain things before inferring conclusions, then how does it affect the validity of the conclusions if the original assumptions are not entirely true? If the assumptions were not entirely true, then would it imply the conclusion to be false? Or would that be considered as an exception to the general rule?

## If not true, then false?

Finding an answer for every question in mathematics may be an impossible task despite utilizing the finite set of axioms and the abstract language of the subject. Most mathematics questions in pre-tertiary school contexts often present themselves into the polarity of right or wrong, correct or incorrect, true or false, *et cetera*. However, in the absence of correctness in the answer of a given mathematics question, does it immediately imply that it is incorrect? In other words, is incorrectness in mathematics the same as being wrong in mathematics? Similarly, how does inaccuracy in mathematics fit into the discussion of incorrectness and wrongfulness?

Topic 1 in the Higher Level programme introduces the notion of proofs, and in particular, proof by mathematical induction. Indeed, mathematical proofs are essential for new conjectures to be proven and their validity accepted. However, is it right to claim that we gain new knowledge in mathematics if a given proof is mathematically valid? If so, then is it necessary for everything in mathematics to be proven true first before one can use it? If not, then how do we distinguish those concepts which are infallible without proofs and those which are proven true, subject to the validity of the proof? More importantly, the foremost critical question is to ask what is considered to be the validity of a proof? If a proof is shown to be true, then does it automatically infer its validity?

One important aspect of mathematical proofs is to provide generalizations of a result. The process of moving from specific results to a generalization is definitely an art. However, have you ever wondered about the potential risk in this process of generalization? How rigourous does one need to be in order to ensure the generalization is not over simplifying the result? If a phenomenon exists with absolute certainty, then what parameters must be established before it could be generalized with symbols and axioms?

Similar to other subject areas, even when a certain knowledge claim has been proven to be true today, no one can guarantee its validity will withstand challenges through future times. Even though it only takes one example to disprove a certain conjecture or theorem, it involves more effort and time than one could ever imagine. Take geometry as an example, what is the shortest distance between two distinct points? In most primary and secondary mathematics classes, the shortest distance between two distinct points is a straight path connecting them; and this is certainly true according to Euclidean geometry. This is Euclid's fifth postulate which dates back to 330 BC.

> *If a straight line crossing two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if extended indefinitely, meet on that side on which are the angles less than the two right angles.*

Euclidean geometry remained unchallenged until the early 19th century when non-Euclidean geometric concepts started to emerge. In Riemannian Geometry and in hyperbolic geometry, the shortest distance between two points is no longer necessarily a straight path. It is often not until the final year of secondary education or even in tertiary education when these challenges are presented and begin to question the first set of knowledge. When situations like this emerge in mathematics (or even just within secondary mathematics education), does it suggest that the content of primary and secondary mathematics is inadequate to enable students to appreciate the fullness of the discipline? Similarly, how incorrect was the first introduction of a straight path being the shortest distance in the classroom? Have you ever questioned whether or not it is acceptable to present some not entirely true statements for the sake of simplifying a complex discussion in mathematics?

Many students and the general public often see mathematics as a subject which has its strength in the provision of absolute certainty. However, in statistics the results are also presented with a tolerance level of uncertainty. How does it affect the validity of the result when it may not necessarily be certain? Conversely, if a given result is presented with 100% certainty, does it mean it is less valid without a certain level of deviation and significance level?

## Imaginary Numbers

An imaginary number is a complex number with a real number and an imaginary unit. However, does the term imaginary number suggest that it is merely just an invention to satisfy the desire of mathematicians? In other words, when the given limits for real numbers do not address the additional mathematical concepts, have the mathematicians created this new concept to fill in the gap? The concept of imaginary numbers was not widely used and adopted until the 18th century by Leonhard Euler and Carl Gauss. When mathematicians cannot find known concepts and existing knowledge to address a new mathematical phenomenon, does it give them an automatic pass to create new sets of rules to govern their findings?

When there is a need to define new number systems, new mathematical rules and theorems, or new methods in approaching emerging topics in mathematics, does it imply that the existing set of axioms is obsolete and inadequate to meet the new demands? If it is necessary for newer rules, is it better to simply create new ones as extensions of the current system, or is it better to start from scratch and disregard all existing rules? If the mathematical field continues to build extensions from the existing set of axioms and rules to facilitate new findings, will there become a time when it becomes impossible to extend any further?

The French have the rather endearing practice of memorialising their great mathematicians. This plaque to the memory of Pierre-Simon, Marquis de Laplace (1749 – 1827) is on the wall of the Central Post Office in Saigon, Vietnam.

# SECTION TWO

# FUNCTIONS

# B.5 Factor and Remainder Theorem

**AHL 2.12**

This chapter will deal with three important results and the ways in which they can help us sketch the graphs of polynomials.

Polynomials are functions of the form:

$$P(x)=a_0+a_1x+a_2x^2+...+a_{n-1}x^{n-1}+a_nx^n$$

### Remainder Theorem

If a polynomial $P(x)$ is divided by a linear polynomial $(x - a)$, the remainder is $P(a)$.

In general: dividend = divisor × quotient + remainder

### Factor Theorem

If, when a polynomial $P(x)$ is divided by a linear polynomial $(x - a)$, the remainder $P(a)$ is zero, then $(x - a)$ is a factor of $P(x)$.

### Fundamental Theorem of Algebra

Every polynomial equation of the form $P(z) = 0$, $z \in \mathbb{C}$, of degree $n \in \mathbb{Q}^+$ has at least one complex root.

This has the important result that:

A polynomial $P_n(z) = 0$, $z \in \mathbb{C}$, of degree $n \in \mathbb{Q}^+$, can be expressed as the product of $n$ linear factors and, hence, produce exactly $n$ solutions to the equation $P_n(z) = 0$.

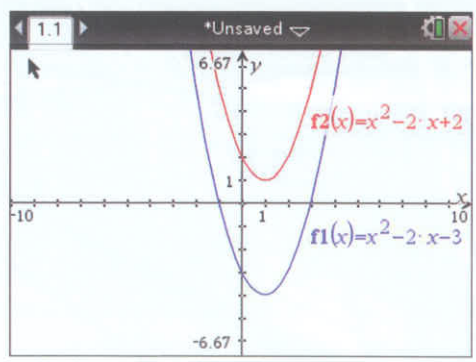
This does not, however, mean that, for example, all cubic equations have three **real** solutions.

The truth of these matters can best be understood by looking at the graphs of polynomials. A good way to do this is with a graphic calculator.

## Polynomial Graphs

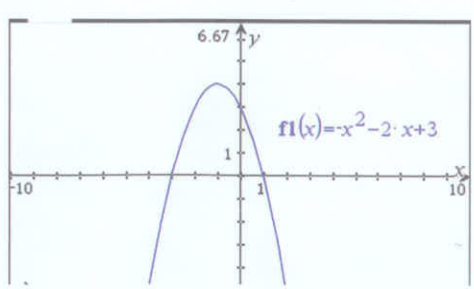
The basic shapes can be investigated using a graphic calculator. Linear functions in which the highest power is 1 are straight lines.

Power 2 polynomials are called quadratics. Their graphs are parabolas. These may or may not intersect the $x$-axis.

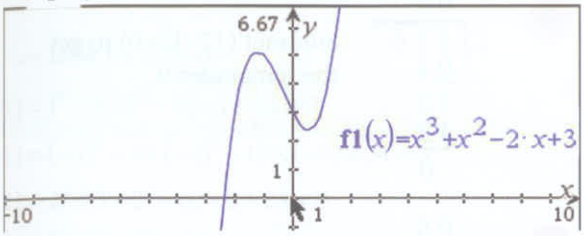



Note that all the principles of graph translation apply to polynomials.

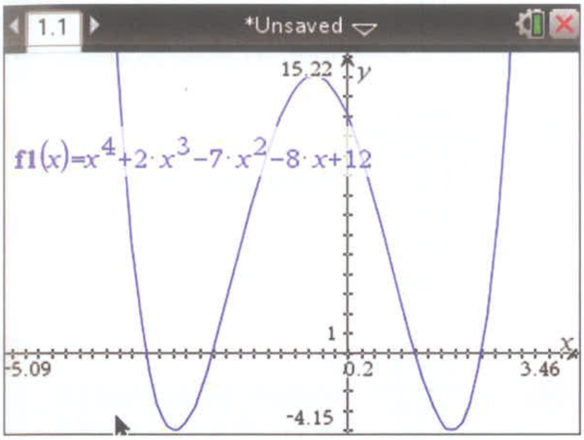
If the squared term is negative, the parabola will be 'inverted'.



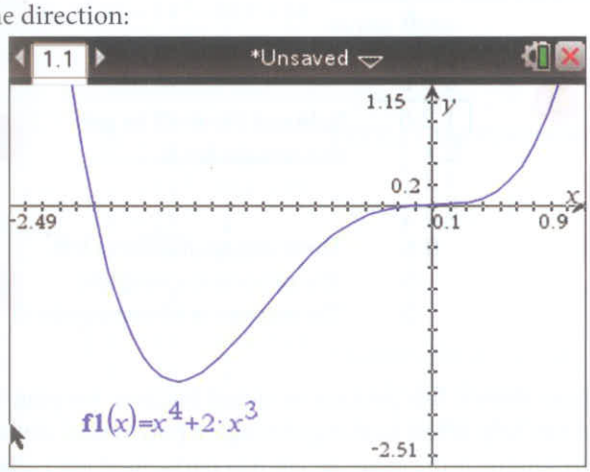
Power 3 polynomials are called cubics.




The extra power has added an extra 'hump' - a maximum or minimum. It seems likely that a power 4 polynomial will have three 'humps' - and this is often true:




There are, however, cases in which a maximum and minimum can coalesce to form a 'point of inflection'. At such points, the graph flattens out for a moment and then carries on in the same direction:




In this case, a maximum and minimum have become an inflection point at the origin.

With that qualification, it is possible to infer the general shape of the graph of a polynomial from the highest power (or degree) of the polynomial.

The other key features are the axes intercepts. The $y$ intercept can be found by evaluating $P(0)$.

The $x$-intercepts are harder as we need to solve $P(x) = 0$. This is usually approached by factorisation using the Factor Theorem.

## Factorising Polynomials

Sometimes we can 'get lucky' with a polynomial and can see how to factorise it.

**Example B.5.1**

Sketch the graph of: $y = x^3 - x^2 - 2x$

The polynomial is cubic. If $x \to \infty$, $y \to +\infty$. Thus, with the qualifications mentioned, we might expect the shape at the top left of this page.

$x = 0, y = 0$ implies the graph passes through the origin.

The other $x$-intercepts must be found by solving $y = 0$.

or $x^3 - x^2 - 2x = 0$.

Since $x$ is a common factor, an immediate factorisation is possible: $x(x^2 - x - 2) = 0$

The quadratic will factorise using the inspection method:

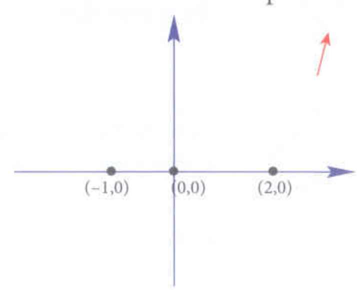
$x(x+1)(x-2) = 0$

Now we use the 'null factor rule' to solve:

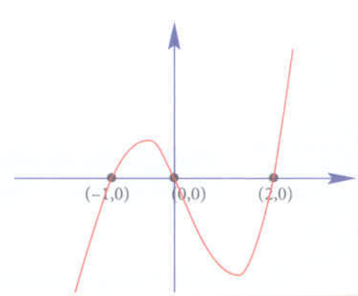
We already have $x = 0$

$x + 1 = 0 \Rightarrow x = -1$ and $x - 2 = 0 \Rightarrow x = 2$.

These features can now be added to a preliminary sketch:




which gives:



If the factorisation is not immediately obvious we use the Factor Theorem.

This works very like the factorisation of numbers. Suppose we were asked to factorise 126? We might begin by noticing that the number is even and so 2 is a factor. Once that is discovered there is a second factor that can be found by the division 126÷2. Many people will do this in their head. However, we will review the process of division as its algebraic version follows an identical pattern.

The common layout for a division of numbers is:

$$2\overline{)\,1\,2\,6}$$

Unlike the other three arithmetic processes (which move from right to left or from small numbers to large), division works from the large numbers to the small, or left to right.

This is because division is sharing. If this problem was to share $126 between 2 people, we would probably begin by sharing the hundred dollar notes out first - which we cannot do as there is not enough money. Next we view it as 12 $10 notes - so each person gets $60. This leaves $6 to share giving the answer as $63.

This is usually written as:

1.
```
   0
2 |1 2 6
```
Divide 1 by 2 (=0).

2.
```
   0
2 |1 2 6
   0
```
Multiply the dividend (0) by the divisor (2) to get 0.

3.
```
   0
2 |1 2 6
   0
   1
```
Subtract (1–0=1) to get the remainder 1.

4.
```
   0
2 |1 2 6
   0↓
   1 2
```
Include the next column to the right - "bring down".

REPEAT the 4 processes.

1.
```
   0 6
2 |1 2 6
   0↓
   1 2
```
Divide 12 by 2 (=6).

2.
```
   0 6
2 |1 2 6
   0↓
   1 2
   1 2
```
Multiply the dividend (6) by the divisor (2) to get 12.

3.
```
   0 6
2 |1 2 6
   0↓
   1 2
   1 2
     0
```
Subtract (12–12=0) to get the remainder 0.

4.
```
   0 6
2 |1 2 6
   0↓
   1 2  |
   1 2  ↓
     0 6
```
Include the next column to the right - "bring down".

REPEAT these processes.

1.
```
   0 6 3
2 |1 2 6
   0↓
   1 2  |
   1 2  ↓
     0 6
```
Divide 6 by 2 (=3).

2.
```
   0 6 3
2 |1 2 6
   0↓
   1 2  |
   1 2  ↓
     0 6
       6
```
Multiply the dividend (3) by the divisor (2) to get 6.

3.
```
   0 6 3
2 |1 2 6
   0↓
   1 2  |
   1 2  ↓
     0 6
       6
       0
```
Subtract (6–6=0) to get the remainder 0.

There are no numbers left.
The process is complete.
The answer is 63 remainder 0

We have shown this process in detail because keeping it in mind can help when working through a polynomial division. The technique is the same except it is performed with algebra instead of arithmetic.

**Example B.5.2**

Factorise: $x^3-3x^2-10x+24$.

Hence sketch the graph of $y=x^3-3x^2-10x+24$.

There is no obvious common factor, so we use the Factor Theorem.

Let: $P(x)=x^3-3x^2-10x+24$

First, we look for a zero:

$$P(1)=1^3-3\times1^2-10\times1+24\neq0$$

$$P(-1)=(-1)^3-3\times(-1)^2-10\times(-1)+24\neq0$$

$$P(2)=2^3-3\times2^2-10\times2+24\neq0$$
$$=8-12-20+24$$
$$=0$$

By the Factor Theorem, $x-2$ is a factor of $P(x)$.

The other factor can be found by division: $P(x)\div(x-2)$.

**1**

$$\begin{array}{r l} & x^2 \\ x-2 & \overline{)\,x^3-3x^2-10x+24} \end{array}$$

Divide $x^3$ by $x$ $(=x^2)$. Only look at the highest powers.

**2**

$$\begin{array}{r l} & x^2 \\ x-2 & \overline{)\,x^3-3x^2-10x+24} \\ & x^3-2x^2 \end{array}$$

Multiply the dividend $(x^2)$ by the divisor $(x-2)$ to get $x^3-2x^2$.

**3**

$$\begin{array}{r l} & x^2 \\ x-2 & \overline{)\,x^3-3x^2-10x+24} \\ & x^3-\underline{2x^2} \\ & \quad -x^2 \end{array}$$

Subtract $(x^3-3x^2-(x^3-2x^2)=-x^2)$ to get the remainder $-x^2$. Take care with signs!

**4**

$$\begin{array}{r l} & x^2 \\ x-2 & \overline{)\,x^3-3x^2-10x+24} \\ & x^3-\underline{2x^2}\ \downarrow \\ & \quad -x^2-10x \end{array}$$

Include the next column to the right - "bring down".

**1**

$$\begin{array}{r l} & x^2-x \\ x-2 & \overline{)\,x^3-3x^2-10x+24} \\ & x^3-\underline{2x^2}\ \downarrow \\ & \quad -x^2-10x \end{array}$$

REPEAT the 4 processes.

Divide $-x^2$ by $x$ $(=-x)$.

**2**

$$\begin{array}{r l} & x^2-x \\ x-2 & \overline{)\,x^3-3x^2-10x+24} \\ & x^3-\underline{2x^2}\ \downarrow \\ & \quad -x^2-10x \\ & \quad -x^2+2x \end{array}$$

Multiply the dividend $(-x)$ by the divisor $(x-2)$ to get $-x^2+2x$.

**3**

$$\begin{array}{r l} & x^2-x \\ x-2 & \overline{)\,x^3-3x^2-10x+24} \\ & x^3-\underline{2x^2}\ \downarrow \\ & \quad -x^2-10x \\ & \quad \underline{-x^2+2x} \\ & \qquad -12x \end{array}$$

Subtract $(-x^2-10x-(-x^2+2x)$ $=-12x)$ to get the remainder $-12x$.

**4**

$$\begin{array}{r l} & x^2-x \\ x-2 & \overline{)\,x^3-3x^2-10x+24} \\ & x^3-\underline{2x^2}\ \downarrow \\ & \quad -x^2-10x \\ & \quad \underline{-x^2+2x}\ \downarrow \\ & \qquad -12x+24 \end{array}$$

Include the next column to the right - "bring down".

**1**

$$\begin{array}{r l} & x^2-x-12 \\ x-2 & \overline{)\,x^3-3x^2-10x+24} \\ & x^3-\underline{2x^2}\ \downarrow \\ & \quad -x^2-10x \\ & \quad \underline{-x^2+2x}\ \downarrow \\ & \qquad -12x+24 \end{array}$$

REPEAT these processes.

Divide $-12x$ by $x$ $(=-12)$.

**2**

$$\begin{array}{r l} & x^2-x-12 \\ x-2 & \overline{)\,x^3-3x^2-10x+24} \\ & x^3-\underline{2x^2}\ \downarrow \\ & \quad -x^2-10x \\ & \quad \underline{-x^2+2x}\ \downarrow \\ & \qquad -12x+24 \\ & \qquad -12x+24 \end{array}$$

Multiply the dividend $(-12)$ by the divisor $(x-2)$ to get $-12x+24$.

**3**

$$\begin{array}{r l} & x^2-x-12 \\ x-2 & \overline{)\,x^3-3x^2-10x+24} \\ & x^3-\underline{2x^2}\ \downarrow \\ & \quad -x^2-10x \\ & \quad \underline{-x^2+2x}\ \downarrow \\ & \qquad -12x+24 \\ & \qquad \underline{-12x+24} \\ & \qquad\qquad 0 \end{array}$$

Subtract $(-12x+24-(-12x+24)=0)$ to get the remainder $0$.

This means that: $P(x)=(x-2)(x^2-x-12)$

The quadratic factor can now be factorised as a trinomial:

$$P(x)=(x-2)(x+3)(x-4)$$

We can now set about sketching the graph of:

$$y=x^3-3x^2-10x+24$$

If $x=0$, $y=24$.

If $y=0$, $x^3-3x^2-10x+24=0$

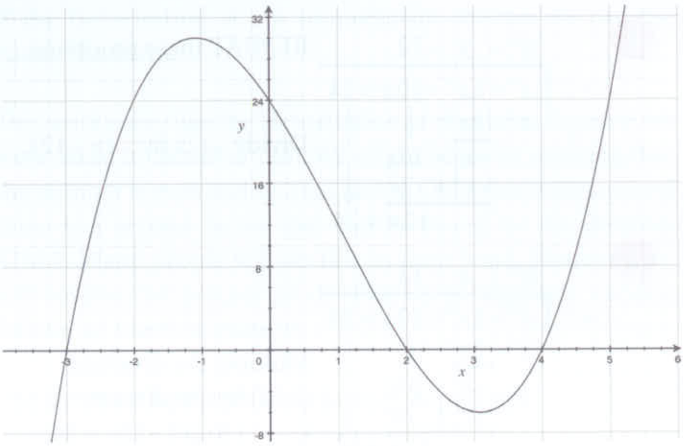
$$(x-2)(x+3)(x-4)=0$$
$$x-2=0\Rightarrow x=2$$
$$x+3=0\Rightarrow x=-3$$
$$x-4=0\Rightarrow x=4$$

We now have all the intercepts: (0,24), (–3,0), (2,0) & (4,0)

Once these four points are on the graph, and with the general shape of the cubic in mind, the sketch can be completed.

An alternative method (synthetic division) is discussed at:

What follows is a computer derived plot. Note that the 'humps' are not symmetric.



### Example B.5.3

Factorise: $P(x)=x^3-2x^2-5x+6$.

Hence sketch the graph of $y=x^3-2x^2-5x+6$.

Using the Factor Theorem:

$$\begin{aligned}P(1)&=1^3-2\times1^2-5\times1+6\\&=1-2-5+6\\&=0\end{aligned}$$

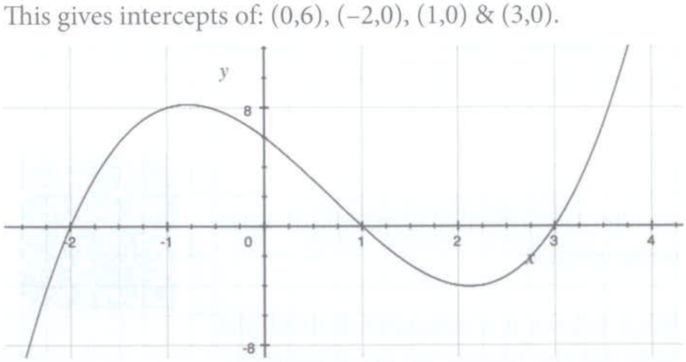
By the Factor Theorem, $x-1$ is a factor of $P(x)$. The other factor can be found by division: $P(x)\div(x-1)$. The required division should look like this:

$$\begin{array}{r|l}
 & x^2 - x \quad - 6 \\ \hline
x-1 & x^3-2x^2-5x+6 \\
 & x^3-x^2 \\ \hline
 & \quad -x^2-5x \\
 & \quad -x^2+x \\ \hline
 & \qquad -6x+6 \\
 & \qquad -6x+6 \\ \hline
 & \qquad\qquad 0
\end{array}$$

So:

$$\begin{aligned}P(x)&=x^3-2x^2-5x+6\\&=(x-1)(x^2-x-6)\\&=(x-1)(x-3)(x+2)\end{aligned}$$

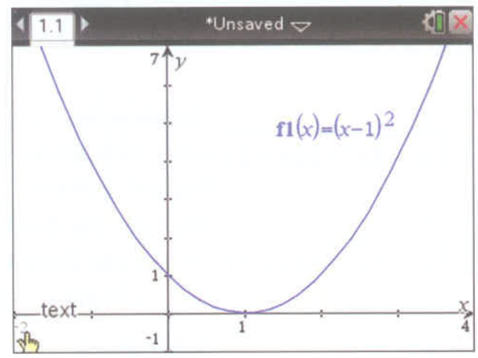
This gives intercepts of: (0,6), (−2,0), (1,0) & (3,0).



## Repeated Factors

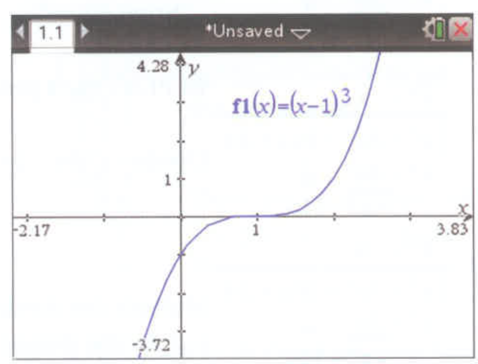
We have a further complication when trying to link the degree of a polynomial with the number of zeros (solutions of $P(x)=0$ and hence the number of intercepts on the graph of $y=P(x)$.

This is illustrated by the graph of $y=(x-1)^2$. This should have two intercepts as it is of degree 2. However, this is not the case. As using a graphic calculator is a good way of understanding this issue, we will use screen grabs in this section.




There is an intercept at (1,0), but it is a 'toucher'. The graph touches, but does not cut, the $x$-axis.

What if the power of the repeated factor is 3?




This time, the intercept is an inflection point and an intercept all in one. It is a good idea to use a calculator to see what happens if the factor is repeated even more times.

The even powers give **touching intercepts** with the graph getting flatter the higher the power.

The odd powers give **inflection intercepts** with the graph getting flatter the higher the power.

With a Casio model, use the Dyna Graph Module (6).

E.g. plot $y=(x-1)^A$.

**Example B.5.4**

Sketch the graph of: $y=(x+1)^2(x-1)^3(x-3)$

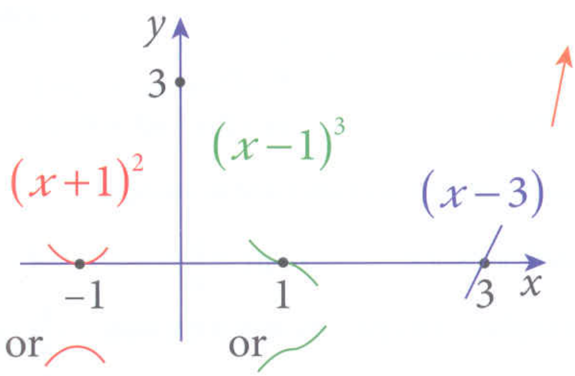
$$y=(x+1)^2(x-1)^3(x-3)$$

The red term (repeated twice) gives a touching intercept at (-1,0).

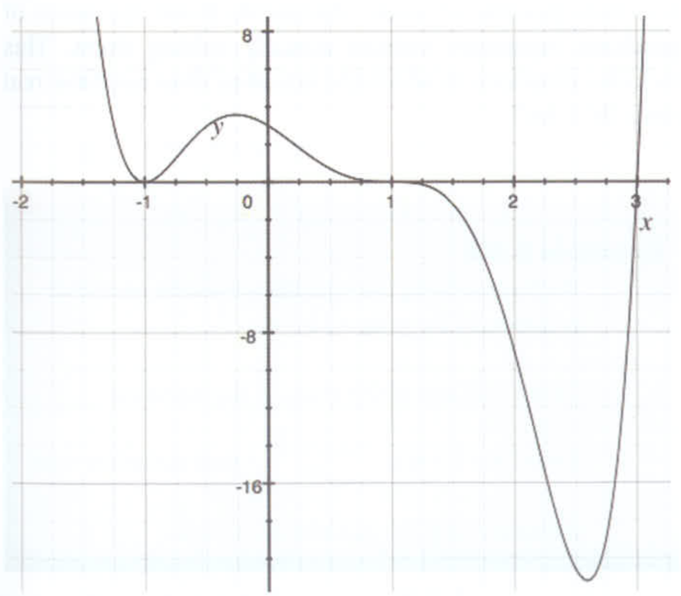
The green term (repeated three times) gives an inflection intercept at (1,0).

The blue term (one only) gives a cutting intercept at (3,0).

The $y$-intercept ($x = 0$) is (0,3). If $x \to \infty$, $y \to +\infty$.




The actual graph is:




## Rational Root Theorem

When given a higher degree polynomial, it is not always easy to find the first zero to begin factorisation. Instead of randomly selecting a value from the real number system, you can consider using the Rational Root Theorem to identify a subset of potential zeros for the given polynomial.

The **Rational Root Theorem** states that if

$$f(x)=a_nx^n+a_{n-1}x^{n-1}+\ldots+a_2x^2+a_1x^1+a_0$$

is a polynomial of degree $n$, then the subset of potential zeros of this given polynomial is $\frac{p}{q}$ where $p$ is the list of the integer factors of $a_0$ and $q$ is the list of factors of $a_n$.

**Example B.5.5**

State all the possible rational roots of:

$f(x)=x^4-8x^3+3x^2+40x-12$

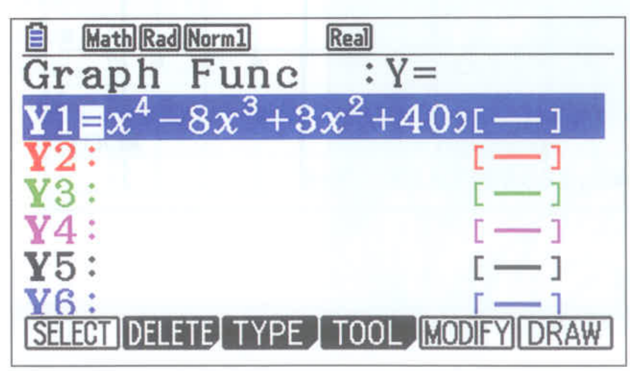
$a_0 = -12$ so the list of factors $p$ is: $\{\pm1,\pm2,\pm3,\pm4,\pm6,\pm12\}$.

$a_n = 1$ so the list of factors $q$ is: $\{\pm1\}$.

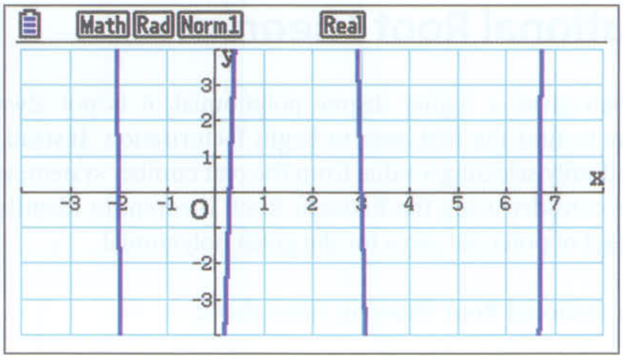
Hence, the list of potential rational roots is $\pm\frac{1,2,3,4,6,12}{1}$.

This will cut down the amount of 'trial and error' involved in using the Factor Theorem.

The other way of doing that is to use a calculatore to draw the graph:

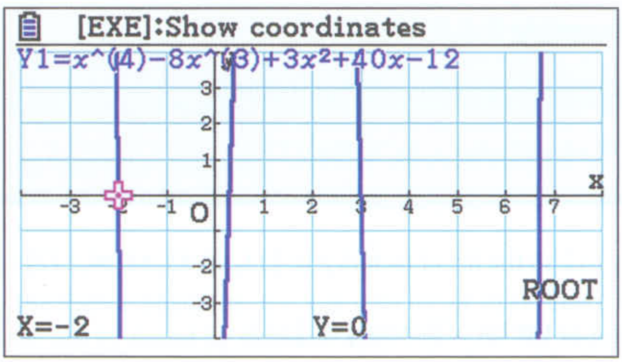



Always remember that you may need to adjust the viewing window. Don't sit looking at your calculator wondering why it "isn't working" when all that has happened is that the graph is off the screen!

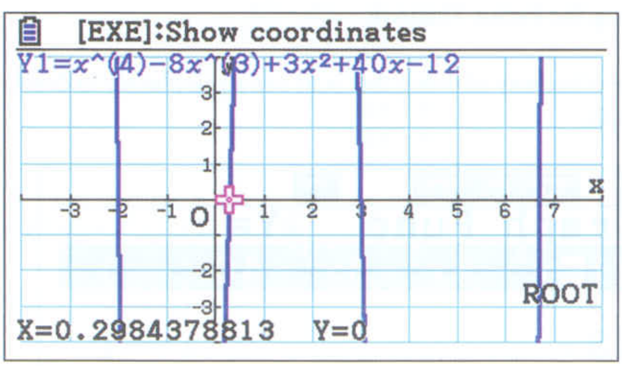



This suggests that –2 is an intercept and hence $(x + 2)$ is a factor of the polynomial.

If necessary, use G-Solve, or Analyse Graph etc. to identify the roots more precisely:




In this case, two of the roots appear to be non-integer.




## Sum and Product of the Roots of a Quadratic Equation

The quadratic equation $ax^2 + bx + c = 0$ can be rearranged to

$$x^2 + \frac{bx}{a} + \frac{c}{a} = 0.$$

Note that the case $a = 0$ can be discounted as it refers to a linear equation.

Assuming the equation will factorise to $(x-\alpha)(x-\beta) = 0$ we can use the null factor rule to identify the roots of the equation as $\alpha$ and $\beta$.

Expanding $(x-\alpha)(x-\beta) = 0$ gives $x^2 - (\alpha+\beta)x + \alpha\beta = 0$.

If this is to be identical to the original equation, then the coefficients of each term must be equal.

Original equation: $x^2 + \frac{b}{a} + \frac{c}{a} = 0$

New version: $x^2 - (\alpha+\beta)x + \alpha\beta = 0$

The coefficients of $x^2$ are both 1 and so are equal.

The coefficients of $x$: $-(\alpha+\beta) = \frac{b}{a}$

which implies: $(\alpha+\beta)$ – 'the sum of the roots' $= -\frac{b}{a}$

The constants: $\alpha\beta$ – 'the product of the roots' $= -\frac{b}{a}$

It is thus possible to make statements about the roots of quadratic equations without actually solving them. This includes situations in which the equation does not have real roots ($b^2 < 4ac$).

**Example B.5.6**

a Without solving the equation, find the sum and products of $2x^2 - 4x - 12 = 0$.

b Given that one of the roots of the equation $5x^2 + 7x - 3 = 0$ is $\frac{-7-\sqrt{109}}{10}$, find the other root without using the quadratic formula.

a Using the sum of the roots $= -\frac{b}{a}$ and the product of the roots $= \frac{c}{a}$:

Sum of the roots $= -\frac{-4}{2} = 2$

Product of the roots $= \frac{-12}{2} = -6$

b One of the roots is $\frac{-7-\sqrt{109}}{10}$.

If the second root is $\beta$, then, using the sum of the roots:

$$\beta + \frac{-7-\sqrt{109}}{10} = \frac{-7}{5}$$

$$\beta = -\frac{-7-\sqrt{109}}{10} + \frac{-7}{5}$$

$$= \frac{7+\sqrt{109}}{10} - \frac{14}{10}$$

$$= \frac{-7+\sqrt{109}}{10}$$

Note that this confirms what might have been expected from the quadratic formula.

## Exercise B.5.1

1. Find the sum and product of the roots of each of these equations.

a $x^2 + 2x + 4 = 0$

b $x^2 - 3x - 7 = 0$

c $x^2 - 3x - 3 = 0$

d $5x^2 - 7x + 3 = 0$

e $2x^2 + 5x - 3 = 0$

f $-9x^2 + 4x + 2 = 0$

g $3x^2 = 7x - 4$

h $5x^2 + 8x = 13$

i $\frac{4x+1}{4x-1} = 2x$

2. Given one of the roots of each of these equations, find the other.

a $x^2 - 5x + 6 = 0$, $\alpha = 2$

b $x^2 - 1 = 0$, $\alpha = 1$

c $2x^2 - 7x + 3 = 0$, $\alpha = 3$

d $6x^2 + x - 1 = 0$, $\alpha = \frac{1}{3}$

e $9x^2 + 12x = 5$, $\alpha = \frac{1}{3}$

f $10x^2 + 27 = 33x$, $\alpha = 1.5$

## Theory of Knowledge

It is not very often that mathematics texts get to recount a tale of passion and revenge.

However, the search for the solutions of polynomial equations is such an opportunity and we are not going to pass it up.

The solution of polynomial equations started with linear and quadratic equations which were 'cracked' quite early on in the History of Mathematics.

However, cubics and higher orders presented a much tougher set of problems.

Some progress had been made in China by Wang Xiaotong in the 7th century and by the Persian poet and scholar Omar Khayyám (1048–1131), both of whom solved a few cubic equations.

The solution to the general cubic, however, remained elusive.

In Bologna at the beginning of the 16th Century, it had become fashionable for the Universtity to stage problem solving competitions. These were a popular 'spectator sport' and drew large crowds.

Two mathematicians, Antonio Fiore and Niccolò Tartaglia (pictured) claimed success with the cubic.

The gauntlet was thrown down and a showdown was arranged in which cubic equations were to be solved against the clock. Tartaglia won.

At this point, a third player, Gerolamo Cardano (pictured), entered the fray.


Cardano succeeded in persuading Tartaglia to tell him his secret. Tartaglia agreed on the condition that Cardano was not to reveal the method to anyone.

However, Cardano shared the secret with a student Lodovico Ferrari with the result that they extended the method to the general solution of the quartic.

What followed was one of the bitterest disputes in the History of Mathematics.

You can read more about these two colourful individuals at:

and

## Exercise B.5.2

1. Sketch the graphs of the following polynomials:

a $P(x) = x(x-2)(x+2)$

b $P(x) = (x-1)(x-3)(x+2)$

c $T(x) = (2x-1)(x-2)(x+1)$

d $P(x) = \left(\frac{x}{3}-1\right)(x+3)(x-1)$

e $P(x) = (x-2)(3-x)(3x+1)$

f $T(x) = (1-3x)(2-x)(2x+1)$

g $P(x) = -x^2(x-4)$

h $P(x) = (1-4x^2)(2x-1)$

i $T(x) = (x-1)(x-3)^2$

j $T(x) = \left(1-\frac{x}{2}\right)^2(x+2)$

k $P(x) = x^2(x+1)(2x-3)$

l $P(x) = 4x^2(x-2)^2$

m $P(x) = \frac{1}{2}(x-3)(x+1)(x-2)^2$

n $T(x) = -(x-2)(x+2)^3$

o $P(x) = (x^2-9)(3-x)^2$

p $T(x) = -2x(x-1)(x+3)(x+1)$

q $P(x) = x^4+2x^3-3x^2$

r $T(x) = \frac{1}{4}(4-x)(x+2)^3$

s $T(x) = -x^3(x^2-4)$

t $T(x) = (2x-1)\left(\frac{x}{2}-1\right)(x-1)(1-x)$

2. Sketch the graph of the following polynomials:

a $P(x) = x^3-4x^2-x+4$

b $P(x) = x^3-6x^2+8x$

c $P(x) = 6x^3+19x^2+x-6$

d $P(x) = -x^3 + 12x + 16$

e $P(x) = x^4 - 5x^2 + 4$

f $P(x) = 3x^3 - 6x^2 + 6x - 12$

g $P(x) = -2x^4 + 3x^3 + 3x^2 - 2x$

h $P(x) = 2x^4 - 3x^3 - 9x^2 - x + 3$

i $T(x) = x^4 - 5x^3 + 6x^2 + 4x - 8$

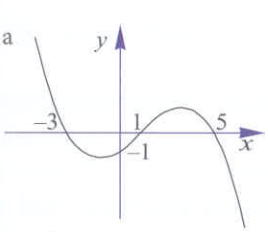
j $T(x) = x^4 + 2x^3 - 3x^2 - 4x + 4$

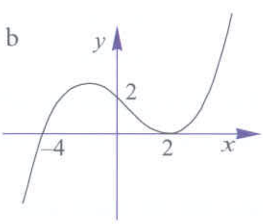
3. Sketch the graphs of:

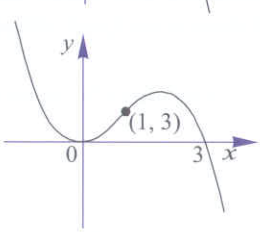
a) $P(x) = x^3 - kx$ where i $k = b^2$ ii $k = -b^2$.

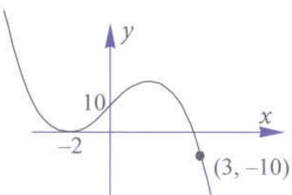
b $P(x) = x^3 - kx^2$ where i $k = b^2$ ii $k = -b^2$.

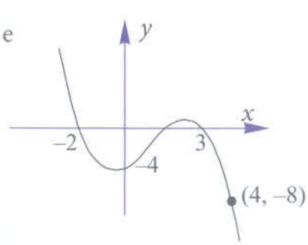
4. Determine the equations of the following cubic functions:

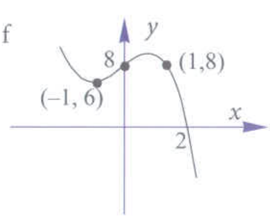
a



b


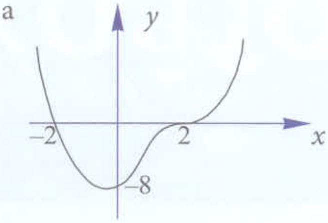

c


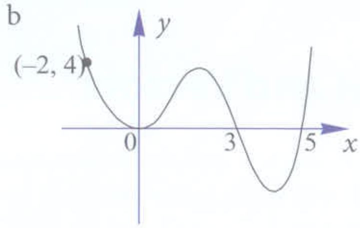

d


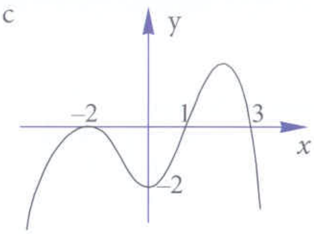

e



f



5. Determine the equation of the following functions:

a



b



c



6. Sketch a graph of $f(x) = (x-b)(ax^2+bx+c)$ if $b > 0$ and:

a $b^2 - 4ac = 0, a > 0, c > 0$

b $b^2 - 4ac > 0, a > 0, c > 0$

c $b^2 - 4ac < 0, a > 0, c > 0$

7. a On the same set of axes sketch the graphs of $f(x) = (x-a)^3$ and $g(x) = (x-a)^2$.

Find $\{(x, y) : f(x) = g(x)\}$.

b Hence find $\{x : (x-a)^3 > (x-a)^2\}$.

Answers


# Functions and Vectors

In the next section, you will study vectors and their properties. This section will only make sense after you have completed Chapter C11.

Vectors are expressions of the type: $\begin{bmatrix} 2 \\ -3 \\ 4 \\ 1 \end{bmatrix}$.

Polynomials are expressions of the type:

$$P(x) = 2x^3 - 3x^2 + 4x + 1$$

We have chosen the coefficients of the polynomial to match that of the vector.

The two items (vector and cubic polynomial) are not the same, but they do have similar forms.

Are the concepts and techniques of vector mathematics applicable to polynomials? This is a fascinating question! We will look as two aspects only.

## 1. Dimension

The dimension of a vector is the number of components.

The vector $\begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix}$ has three components.

It is said to be a three dimensional vector and it can be represented geometrically in our 3-dimensional space. Thus, quadratic polynomials can be thought of as being 3-dimensional as well.

Since there is no limit to the number of terms in a polynomial, the vector space occupied by them is of infinite dimension. What does this mean?

## 2. Angle

The angle between two vectors is related to their scalar product. If the scalar product of two vectors is zero, then they are at right angles to one another.

For example:

$$\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \bullet \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = 2 \times 2 + 1 \times -1 + (-1) \times 3$$
$$= 4 - 1 - 3$$
$$= 0$$

This implies that the vectors $\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$ are at right angles to one another.

But what of the associated polynomials?

Is there any sense in the statement that:

$P(x) = 2x^2 + x - 1$ is at right angles to $Q(x) = 2x^2 - x + 3$ ?

There are some calculus based (definite integral) definitions of scalar product that have proved useful. How do these work?

# B.6 Rational Functions

AHL 2.13

## The Rational Function

Rational functions take the form: $f(x)=\dfrac{p(x)}{q(x)}$. Graphs of this nature possess three types of asymptotes, one vertical another horizontal and thirdly, diagonal.

### 1. The vertical asymptote

Firstly consider functions of the form: $y=\dfrac{ax+b}{cx+d}$

A vertical asymptote occurs when the denominator is zero, that is, where $cx+d=0$. Where this occurs, we place a vertical line (usually dashed), indicating that the curve cannot cross this line under any circumstances. This must be the case, because the function is undefined for that value of $x$.

For example, the function

$$x \mapsto \frac{3x+1}{2x+4}$$

is undefined for that value of $x$ where $2x+4=0$. That is, the function is undefined for $x=-2$. This means that we would need to draw a vertical asymptote at $x=-2$. In this case, we say that the asymptote is defined by the equation $x=-2$.

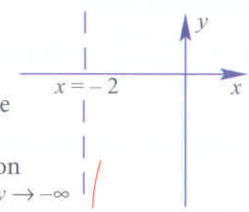
Using limiting arguments provides a more formal approach to 'deriving' the equation of the vertical asymptote. The argument is based along the following lines:

as $x \to -2^-, \dfrac{3x+1}{2x+4} \to +\infty$

That is, as $x$ tends to $-2$ from the left or 'below', (hence the minus sign next to the two) the function tends to positive infinity.


as $x \to -2^+, \dfrac{3x+1}{2x+4} \to +\infty$

That is, as $x$ tends to $-2$ from the right or 'above', (hence the plus sign next to the two) the function tends to negative infinity.




Therefore we write:

$$\left.\begin{array}{l}\text{As } x \to -2^+ \quad f(x) \to -\infty \\ \text{As } x \to -2^- \quad f(x) \to +\infty\end{array}\right\} \begin{array}{l}\therefore x = -2 \\ \text{is a vertical} \\ \text{asymptote}\end{array}$$

of $f(x)=\dfrac{3x+1}{2x+4}, x \neq -2$.

### 2. The horizontal asymptote

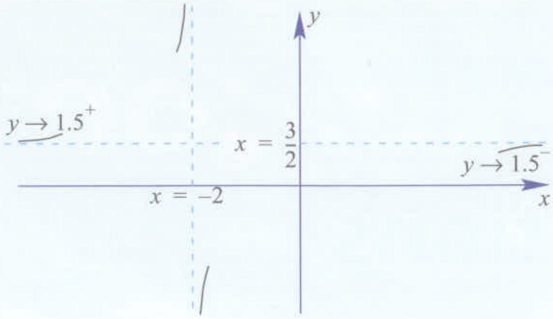
To determine the equation of the horizontal asymptote, we use a limiting argument, however, this time we observe the behaviour of the function as $x \to \pm\infty$.

It will be easier to determine the behaviour of the function (as $x \to \pm\infty$) if we first 'simplify' the rational function (using long division):

$$f(x) = \frac{3x+1}{2x+4} = \frac{3}{2} - \frac{5}{2x+4}.$$

Next we determine the behaviour for extreme values of $x$.

$$\left.\begin{array}{l}\text{As } x \to +\infty \quad f(x) \to \left(\tfrac{3}{2}\right)^- \\ \text{As } x \to -\infty \quad f(x) \to \left(\tfrac{3}{2}\right)^+\end{array}\right\} \begin{array}{l}\text{Therefore, } y = \dfrac{3}{2} \\ \text{is the horizontal} \\ \text{asymptote}\end{array}$$




We can now add a few more features of the function:

## 3. Axial intercepts

**$x$–intercept**

To determine the $x$-intercept(s) we need to solve for $f(x) = 0$.

In this case we have:

$$f(x) = \frac{3x+1}{2x+4} = 0 \Leftrightarrow 3x+1 = 0 \Leftrightarrow x = -\frac{1}{3}.$$

That is, the curve passes through the point $\left(-\frac{1}{3}, 0\right)$.

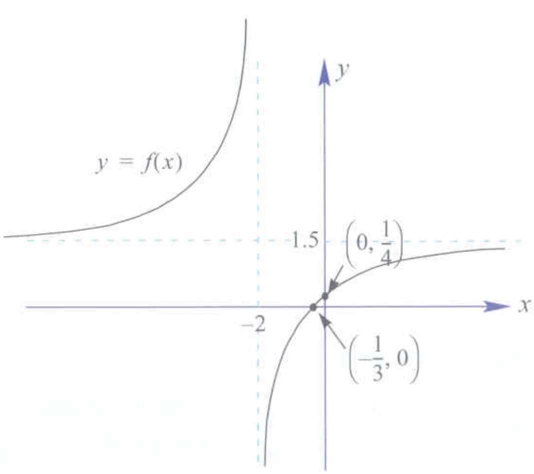
**$y$-intercept**

To determine the $y$-intercept we find the value of $f(0)$ (if it exists, for it could be that the line $x = 0$ is a vertical asymptote).

In this case we have $f(0) = \dfrac{3\times0+1}{2\times0+4} = \dfrac{1}{4}$.

Therefore the curve passes through the point $(0, {}^1/_4)$.

Having determined the behaviour of the curve near its asymptotes (i.e. if the curve approaches the asymptotes from above or below) and the axial–intercept, all that remains is to find the stationary points (if any).




**Example B.6.1**

Sketch the graph of: $y = \dfrac{3x+1}{1-x}$

The vertical asymptote is at $x = 1$. Just to the left of the asymptote (e.g. at $x = -1.1$) the numerator is negative and the denominator small and negative. The $y$ value is large and positive. Just to the right of the asymptote (e.g. at $x = -0.9$) the numerator is negative and the denominator small and positive. The $y$ value is large and negative.

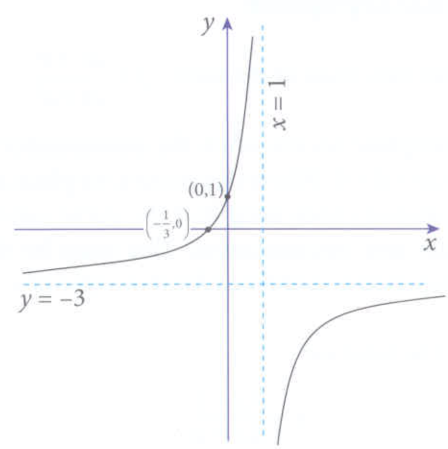
The horizontal asymptote occurs when $x$ is large. When this happens, $y$ tends to $-3$. For a big negative $x$ (e.g. $x = -100$), $y = {}^{-299}/_{101}$ i.e. a bit smaller than $-3$ (above the asymptote). For a big positive $x$ (e.g. $x = 100$), $y = {}^{301}/_{-99}$ i.e. a bit smaller than $-3$ (below the asymptote).

Intercepts:

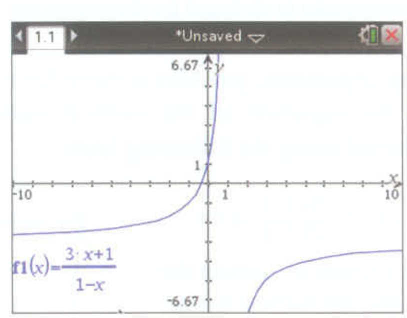
$x$ intercept at $3x+1=0 \Rightarrow x = -\dfrac{1}{3}$

$y$ intercept at $y = \dfrac{3\times0+1}{1-0} = 1$

The sketch should show all the important features:




Modern graphic calculators will produce quite good plots of rational functions. They do not, however, show the key features which a good sketch must include.



### Exercise B.6.1

1. Use a limiting argument to determine the equations of the vertical and horizontal asymptotes for the following.

   a $f(x)=\dfrac{2x+1}{x+1}$ b $f(x)=\dfrac{3x+2}{3x+1}$

   c $f(x)=\dfrac{2x-1}{4x+1}$ d $f(x)=\dfrac{4-x}{x+3}$

   e $f(x)=3-\dfrac{1}{x}$ f $f(x)=5-\dfrac{1}{2-x}$

2, Make use of a graphics calculator to verify your results from Question 1 by sketching the graph of the given functions.

3. Sketch the following curves, clearly labelling all intercepts, stating the equations of all asymptotes, and, in each case, showing that there are no stationary points.

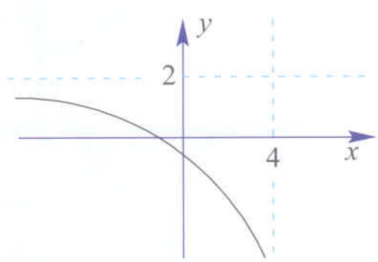
   a $x\mapsto\dfrac{3}{2x+1}$ b $x\mapsto\dfrac{x+1}{x+2}$

   c $x\mapsto\dfrac{5-x}{2x-1}$ d $x\mapsto 3+\dfrac{1}{x}$

   e $x\mapsto\dfrac{1}{x-3}-2$ f $x\mapsto 1-\dfrac{2}{2x-3}$

4. The figure at below shows part of the graph of the function whose equation is:

   $x\mapsto\dfrac{ax+2}{x-c}$ .




   Find the values of $a$ and $c$.

5. Given that $f:x\mapsto x+2$ and that $g:x\mapsto\dfrac{1}{x-1}$ , sketch the graphs of:

   a $f\circ g$ b $g\circ f$.

Extra questions

## Quadratic Numerator

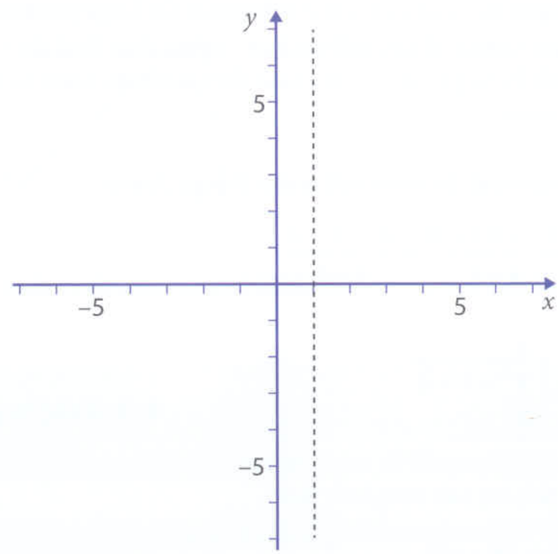
This section will deal with functions of the form:

$$f(x)=\frac{ax^2+bx+c}{dx+e}$$

**Example B.6.2**

Sketch the graph of: $y=\dfrac{(x-2)(x+1)}{x-1}$

As with the examples we have already dealt with, the value of $x$ that gives a zero denominator (1) will give rise to a vertical asymptote.




It can also be helpful to work out the behaviour of the graph just either side of the asymptote - it will either be large and negative or large and positive.

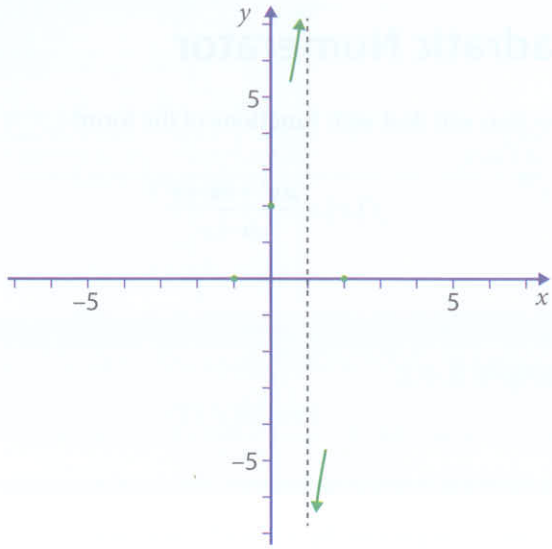
If $x$ is just less than 1: $y=\dfrac{(-)(+)}{-}$ - large and positive.

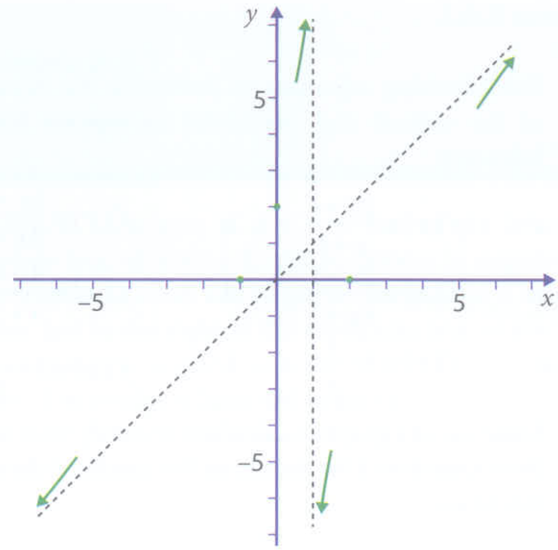
If $x$ is just more than 1: $y=\dfrac{(-)(+)}{+}$ - large and negative.

The zero(s) of the numerator can also be added at this time as these give the $x$-intercepts.

$(x-2)(x+1)=0\Rightarrow x=-1,2$

The $y$-intercept: $y=\dfrac{(0-2)(0+1)}{0-1}=\dfrac{-2\times1}{-1}=2$ .




Whilst this makes the shape of the graph in the vicinity of the origin fairly obvious, the behaviour for large positive and negative values of $x$ is still unclear - other than that the values of $y$ will be large. This is because the quadratic term is in the numerator.

A polynomial division will make things clearer.

$$y=\frac{(x-2)(x+1)}{x-1}=\frac{x^2-x-2}{x-1}$$

$$\begin{array}{r|l} & x \\ x-1 & x^2-x-2 \\ & \underline{x^2-x} \\ & \qquad -2 \end{array}$$

From this we can conclude that:

$$y=\frac{(x-2)(x+1)}{x-1}=x-\frac{2}{x-1}$$

If $x$ gets large (either positive or negative), the remainder term becomes small and the function behaves more and more like $y=x$ . This line can now be added as an oblique asymptote.

If $x$ is large and negative, the remainder term is positive and the graph will be above the asymptote.

If $x$ is large and positive, the remainder term is negative and the graph will be below the asymptote.

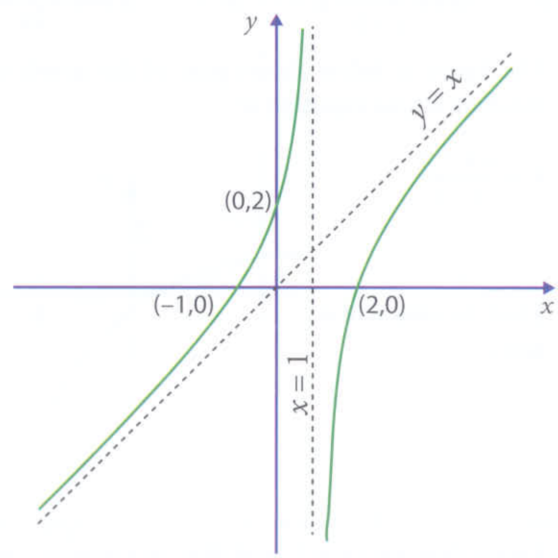
If all this information is transferred to the graph, the shape becomes much clearer.

The final sketch should show all the key information.

This means that it should have the correct **shape**.

The coordinates of the **intercepts** should be given.

The equations of the **asymptotes** should be given.




**Example B.6.3**

Sketch the graph of: $y=\frac{(x-1)(x+1)}{x-2}$

Intercepts:

$x=0$: $y=\frac{(0-1)(0+1)}{0-2}=\frac{1}{2}$

$y=0$: $\frac{(x-1)(x+1)}{x-2}=0 \Rightarrow (x-1)(x+1)=0 \Rightarrow x=-1,1$

Vertical Asymptote:

The denominator is zero: $x = 2$

To the left of the asymptote: $y=\frac{(+)(+)}{-}=-$

To the right of the asymptote: $y=\frac{(+)(+)}{+}=+$

Diagonal Asymptote:

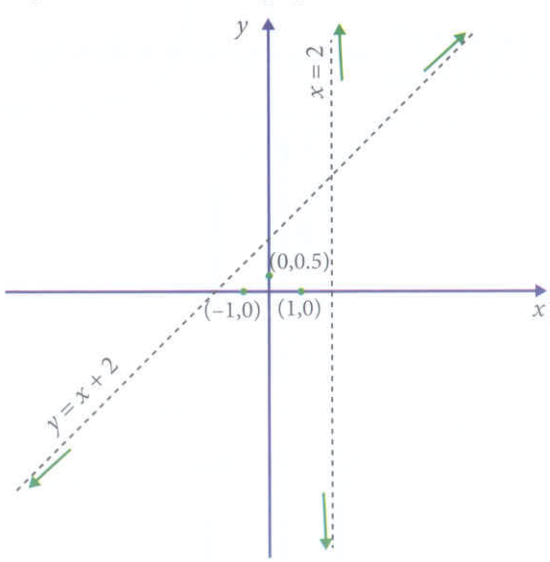
After division: $y=\frac{(x-1)(x+1)}{x-2}=x+2+\frac{3}{x-2}$

There is a diagonal asymptote of $y = x + 2$.

The remainder term is negative for large negative $x$ so the graph is below the asymptote at the left.

The remainder term is positive for large positive $x$ so the graph is above the asymptote at the right.

These key facts translate to a graph:



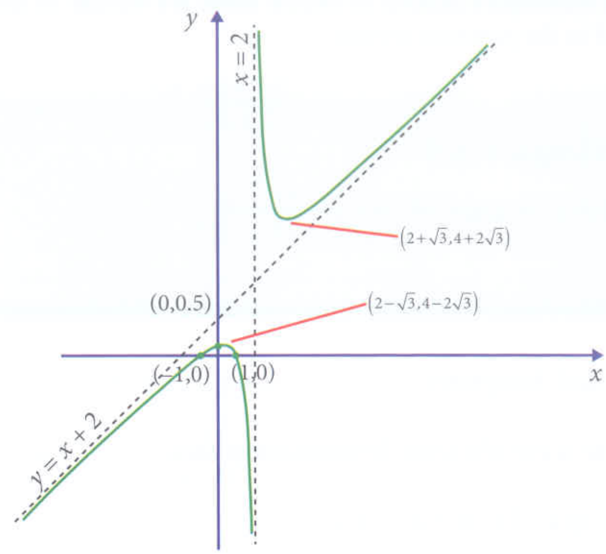

On this occasion, it appears that there are two turning points. These can be found using calculus (the quotient rule):

$$\frac{dy}{dx}=\frac{x^2-4x+1}{(x-2)^2}$$

Equating the numerator to zero: $x=2\pm\sqrt{3}$

The turning points are $\left(2\pm\sqrt{3},4\pm2\sqrt{3}\right)$

or approximately: (0.27, 0.54) and (3.73, 7.46)




**Example B.6.4**

Find the asymptotes and symmetries of the graph of:

$$y=\frac{(2x-3)(1-x)}{x-2}$$

There is a vertical asymptote at $x = 2$ (zero denominator).

Division gives: $y=\frac{(2x-3)(1-x)}{x-2}=-2x+1-\frac{1}{x-2}$

There is a diagonal asymptote $\left(y=-2x+1\right)$ that intersects the vertical asymptote at (2,–3).

The point (2,–3) is a centre of two-fold rotational symmetry.

## Quadratic Denominator

This section will deal with functions of the form:

$$f(x)=\frac{ax+b}{cx^2+dx+e}$$

The techniques needed to sketch these are similar to those used in the previous section.

**Example B.6.5**

Sketch the graph of: $y=\frac{x-3}{(x+1)(x-2)}$

**Vertical Aymptotes:**

These occur when the denominator is zero:

$(x+1)(x-2)=0 \Rightarrow x=-1,2$

To the left of $-1$, $y=\frac{-}{(-)(-)}=-$, to the right $y=\frac{-}{(+)(-)}=+$.

To the left of 2, $y=\frac{-}{(+)(-)}=+$, to the right $y=\frac{-}{(+)(+)}=-$.

**Other Asymptotes:**

Because the numerator is linear and the denominator is quadratic, we expect that, for large $x$ values, $y$ becomes small.

For large negative $x$, $y=\frac{-}{(-)(-)}=-$.

We can expect the graph to approach the $x$-axis from below to the left.

For large positive $x$, $y=\frac{+}{(+)(+)}=+$.

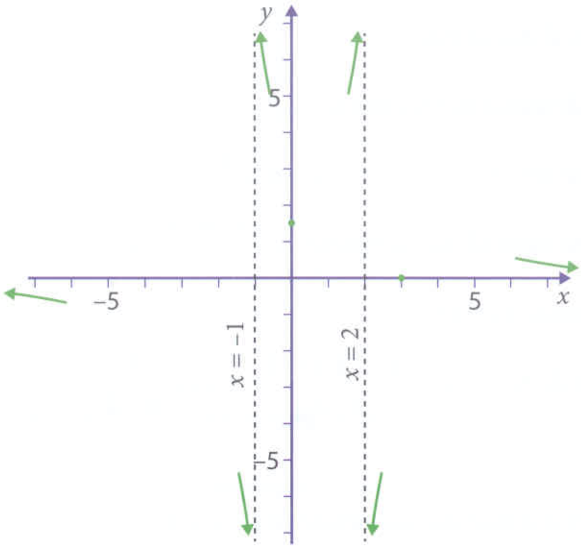
We can expect the graph to approach the $x$-axis from above to the right.

**Intercepts:**

$y=0 \Rightarrow x-3=0 \Rightarrow x=3$

$x=0 \Rightarrow y=\frac{0-3}{(0+1)(0-2)}=\frac{-3}{-2}=1.5$

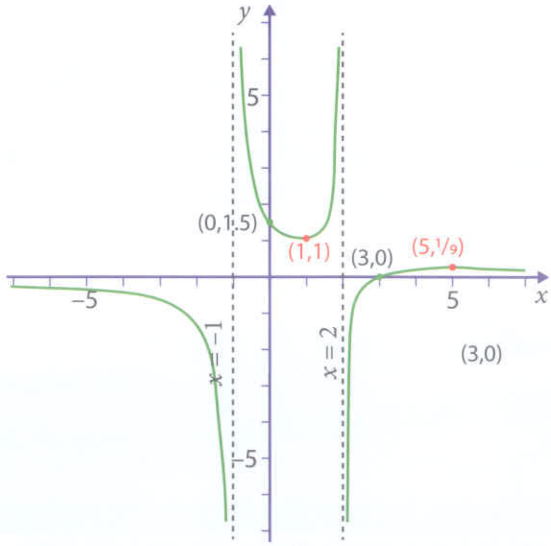
On a set of axes, these are:




It appears that there must be a minimum point between the asymptotes and a maximum point in the region of $x = 4$.

Using the quotient rule: $\frac{dy}{dx}=\frac{-x^2+6x-5}{(x+1)^2(x-2)^2}$

$\frac{dy}{dx}=\frac{-x^2+6x-5}{(x+1)^2(x-2)^2}=0 \Rightarrow x=1,5$ ie. $(1,1)$ & $(5,{}^1/_9)$.



### Example B.6.6

Sketch the graph of: $y=\dfrac{3-x}{x^2-x-6}$

Vertical Asymptotes:

$$\begin{aligned} x^2-x-6&=0\\ (x+2)(x-3)&=0\\ x&=-2,3 \end{aligned}$$

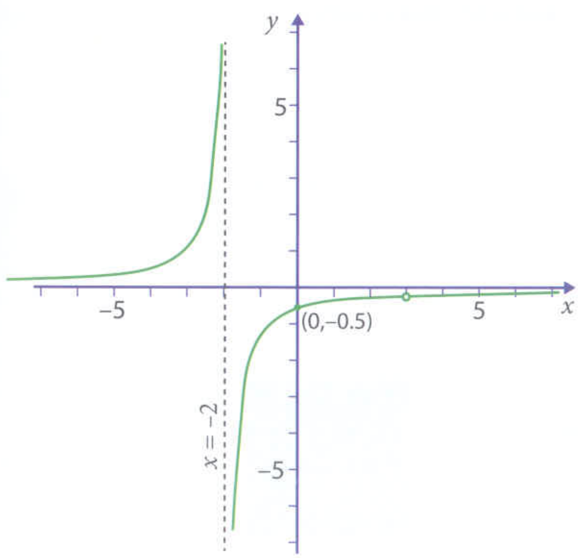
However, for $x = 3$, the numerator is zero as well. There is therefore a 'hole' in the graph at this point and not an asymptote.

For large negative $x$, the graph approaches the $x$-axis from above.

For large positive $x$, the graph approaches the $x$-axis from below.

Intercepts:

$x=0,\ y=\dfrac{3-0}{0^2-0-6}=-\dfrac{1}{2}$




## Exercise B.6.2

1. Sketch the graphs of:

a $y=\dfrac{x(x-1)}{x+2}$

b $y=\dfrac{x^2-1}{x+2}$

c $y=\dfrac{x^2+3x+2}{x-1}$

d $y=\dfrac{(2-x)x}{x+1}$

2. Sketch the graphs of:

a $y=\dfrac{x}{(x+1)^2}$

b $y=\dfrac{x-4}{(x+1)(x-2)}$

c $y=\dfrac{1-2x}{(x+2)(x-1)}$

d $y=\dfrac{3-2x}{6x^2-x-1}$

3. State the equations of the asymptotes of the graphs of these functions:

a $y=\dfrac{x^2-x-3}{x+4}$

b $y=\dfrac{4}{2x^2-x-1}$

c $y=\dfrac{4-x}{x^2+x-2}$

d $y=\dfrac{x^2+x-2}{1-x}$

4. Find the value(s) of $a$ such that the graph of:

$$y=\frac{2x+3}{ax^2-2x+3}$$

has exactly one vertical asymptote.

5. Find the coordinates of the centre of symmetry of the graph of:

$$y=\frac{x^2-3x-4}{x-1}$$

6. Find the value of $a$ such that the graph of:

$$y=\frac{2x+7}{x^2+x+a}$$

has asymptotes $y=0, x=-4, 3$.

Sketch the graph.

7. Find the value of $a$ such that the graph of:

$$y=\frac{x^2+ax+3}{x+3}$$

has $y=x-7$ as an asymptote.

8. Find the value(s) of $a$ such that the graph of:

$$y=\frac{x^2+7x+5}{ax-1}$$

has no vertical asymptote.

9. If $f(x)=\dfrac{x+7}{(x-2)(x+3)}$, sketch the graphs of:

a $y=f(x)$

b $y=|f(x)|$

c $y=f(|x|)$

10. Find the coordinates of the centre of symmetry of the graph of:

$$y=\frac{(x+a)(x-1)}{x-a}$$

11. The unit profit ($\$p$,000) of a new electronic component depends on the weekly output ($n$ thousands).

The expected unit profits are modelled by:

$$p=\frac{58n-30}{(n+4)(n+1)}, n>0$$

Use a graphical method to find the optimum output.

13. The pressure ($p$) at time $t$ after an impact is modelled by:

$$p=\frac{5t+2}{(t-1)(t-7)}+11, 1<t<7$$

Use a graphical method to find the maximum pressure and when this occurs.

Answers

# B.7 Further Functions

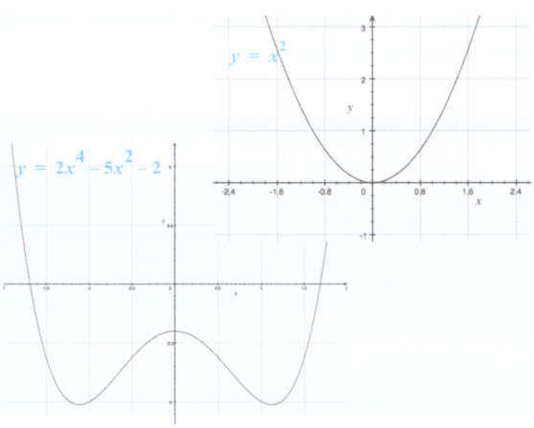
AHL 2.14
AHL 2.15

## Odd and Even Functions

Functions can be classified into three categories based on the symmetries of their graphs.

### Even functions

If a function has line symmetry about the $y$-axis, it is said to be even.



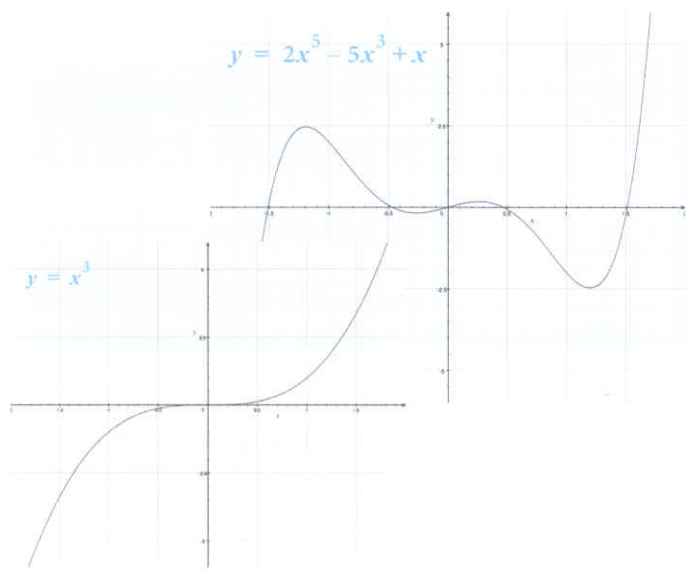

Formally, a function is even if $f(x) = f(-x)$ for all $x$ in the domain.

The most obvious examples of even functions are the even polynomials, $x^2$, $x^4$... Other examples are $|x|$, $\cos(x)$.

### Odd functions

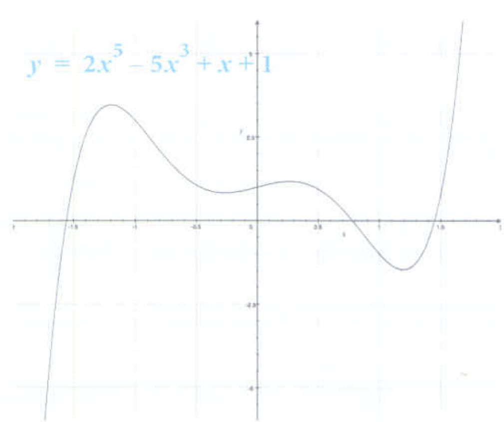
A function is odd if it has two-fold rotational symmetry about the origin. If the graph is 'pinned' at the origin and turned through 180°, it will fit back over the original graph.

Formally, a function is odd if $f(x) = -f(-x)$ for all $x$ in the domain, e.g. $x$, $x^3$, $\sin(x)$.




The concept applies to other, non-polynomial functions. The cosine function is even, the sine function is odd and the logarithm function is neither.

Not every function is either odd or even. Most functions are neither.

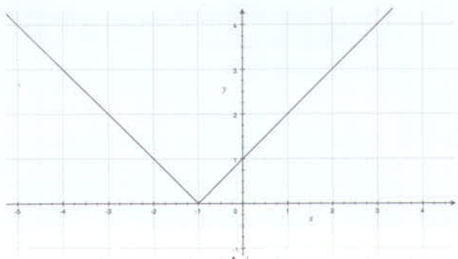


**Example B.7.1**

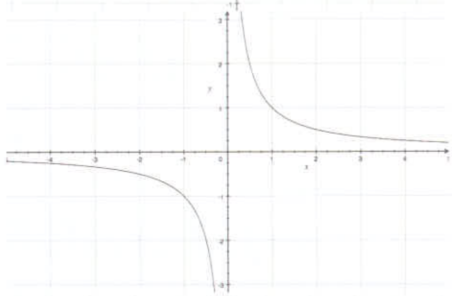
Classify the following functions as even, odd or neither.

a $f(x) = |x+1|$  b $f(x) = \frac{1}{x}$  c $f(x) = \left|\frac{1}{x}\right|$

a  Neither odd nor even.



b  Odd



c  Even

### Exercise B.7.1

1. Classify the following functions as even, odd or neither.

   a $y = 4$  b $y = |x^3|$

   c $y = (x-1)^3$  d $y = \frac{x}{x-2}$

   e $y = \ln(x^2)$  f $y = \frac{1}{x}$

   g $y = \frac{x^2+2}{x^3+x}$  h $y = \frac{x}{x^3-x}$

   i $y = \frac{\sin x}{\cos x} + x^3$

2. Prove that the product of two even functions is even.

3. Is it necessary for $f(0)=0$ for a function to be odd?

4. Explain why the composite of two odd functions is odd.

5. Prove that the quotient of two even functions is even.

6. A function has the full real line as its domain and is both odd and even. What is the rule for the function?

## Identity and inverse functions

The main stages of finding an inverse function were covered in Chapter B2 of the SL text.

We begin with an example to review the key techniques.

**Example B.7.2**

Find the inverse function of $f(x)=\frac{2x-3}{5}, x \in \mathbb{R}$

The first step is to write this as a '$y =$' statement.

$$y=\frac{2x-3}{5}$$

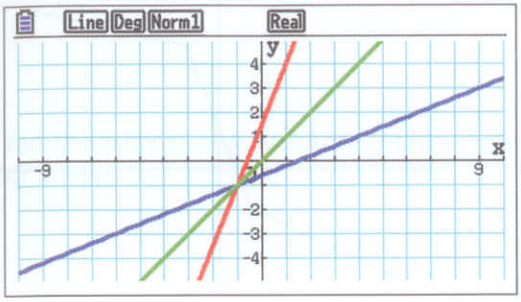
Next, invert the function by exchanging $x$ & $y$.

$$x=\frac{2y-3}{5}$$

Then, make $y$ the subject:

$$5x=2y-3$$
$$2y=5x+3$$
$$y=\frac{5x+3}{2}$$
$$f^{-1}(x)=\frac{5x+3}{2}, x\in\mathbb{R}$$

It is a good idea to check that the answer is correct graphically.




The original function is blue, the inverse is red and $y = x$ is the symmetry line between a function and its inverse. There is also a common point between the three lines.

If using a graphic calculator it is a good idea to use one of the 'square' options for the axes - this makes the symmetry line at 45° to the axes.

The domains of both the function and its inverse are the full real line.

Finding inverses is seldom as simple as the previous example. There are two problems. The first is that the inverse of many to one functions is not a function and the other relates to domains.

### Example B.7.3

Find the inverse function of $f(x)=(x-4)^2+2, x\in\mathbb{R}$.

$y=(x-4)^2+2$, so the inverse is:

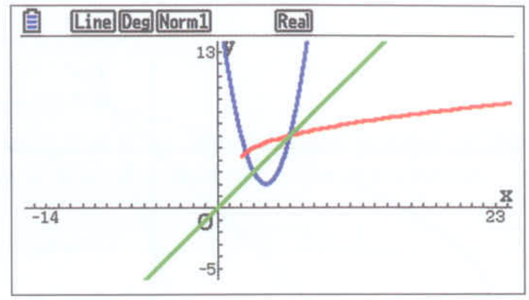
$$x=(y-4)^2+2$$
$$x-2=(y-4)^2$$
$$\sqrt{x-2}=y-4$$
$$y=+\sqrt{x-2}+4$$

The positive sign in front of the square root indicates that the negative option of the square root is excluded.

This converts the expression to a function.

The issue of domain also relates to the fact that we cannot have a negative value inside the square root. It follows that the domain of the inverse is $x \geq 2$.

Graphically, this is:




If the original function is to have an inverse its domain has to be restricted to the right hand 'limb' of the curve:

$$f(x)=(x-4)^2+2, x\geq 4$$
$$f^{-1}(x)=+\sqrt{x-2}+4, x\geq 2$$

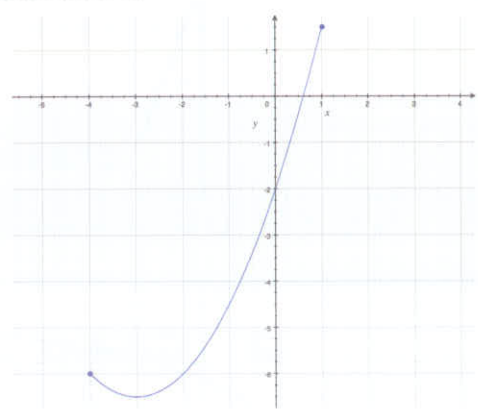
### Example B.7.4

Find the maximal domain of the function:

$$f(x)=\frac{x^2}{2}+3x-2, x\in[-4,1]$$

such that it has a well defined inverse.

The question does not ask for the actual rule of the inverse, so it can be tackled graphically. The graph of $f$, including the domain restriction is:

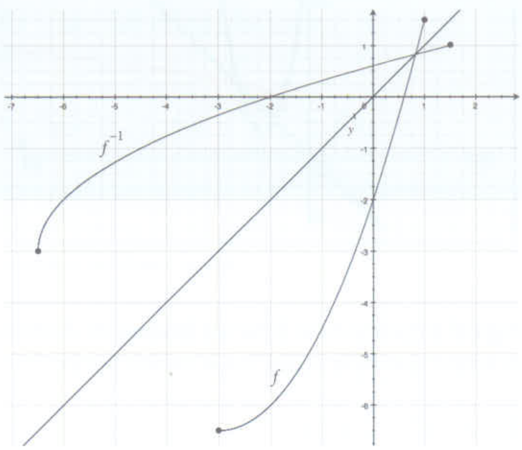


The inverse will only be a function if it is either one:one or many:one. We therefore need to look for parts of the graph of the original function where it is one:one. This happens either to the left of the minimum point or to its right. The minimum point is (–3,–6.5) - use calculus if necessary.

This can be achieved either by choosing the part for which:

$f:[-4,-3]\to[-6.5,-6]$ or $f:[-3,1]\to[-6.5,1.5]$.

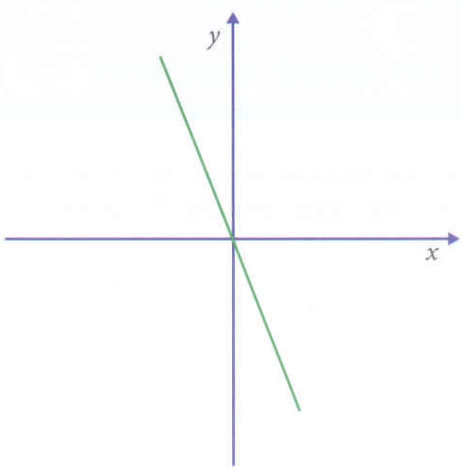
The second of these options is graphically:



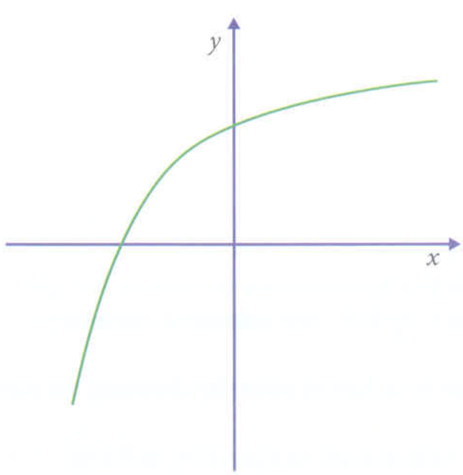

## Exercise B.7.2

1. Sketch the graphs of the inverses of these functions:

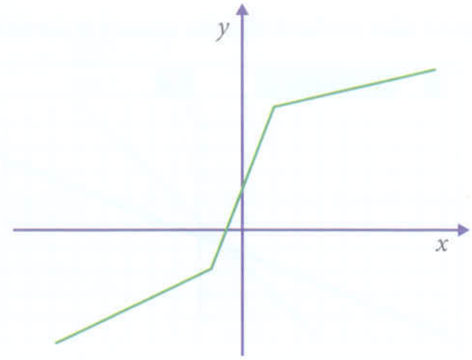
a




b

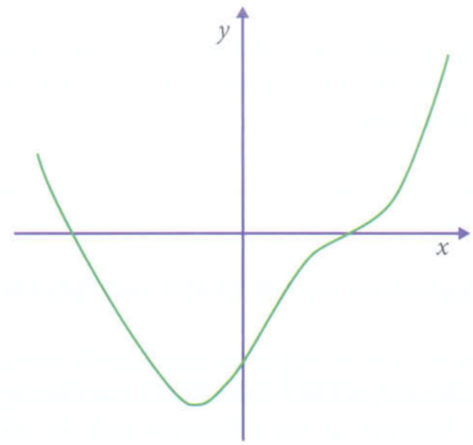



c

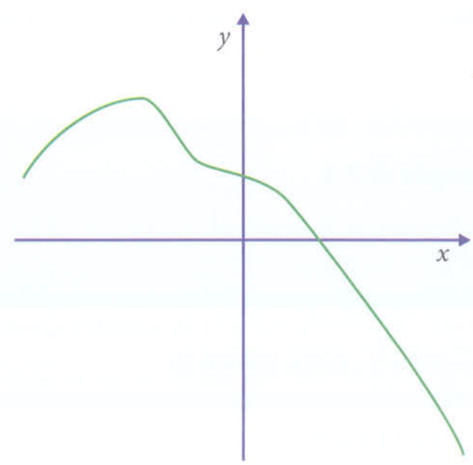



2. Restrict the domains of these functions so that the inverse is a function and show the result as a graph.

a




b




3. Find the rule for the inverse of:

$f(x)=\sqrt{x-1}, x \geq 1$ stating its domain.

4. Find the value of $a$ such that function:

$f(x)=x^2-4x+7, x \in [a,20]$ has a well defined inverse.

5. If $f(x)=e^{2x-1}, x \in \mathbb{R}$, find the inverse function, stating any domain restrictions.

6. If $f(x)=2^{x^2}-3, x\in\mathbb{R}$, find the function $g$ such that $fog=x$.

7. Find the value(s) of $p$ such that $f(x)=\frac{p}{x^p}, x>0$ is self inverse.

8. The function:

$$f(x)=\sin\left(\frac{\pi x}{4}\right), x\in[-a,a]$$

has a well defined inverse. Find the value of $a$.

## Solving equations

In all cases, if an equation is solvable by an analytic methods, then it is best done that way. But with care!

**Example B.7.5**

If $f(x)=\sqrt{2x-1}, x>1$ and $g(x)=2x-7, x>0$, find the smallest value of $x$ for which $f(x)=g(x)$.

An analytic solution might proceed as follows:

$$\sqrt{2x-1}=2x-7$$
$$2x-1=(2x-7)^2$$
$$2x-1=4x^2-28x+49$$
$$0=4x^2-30x+50$$

Using the quadratic formula:

$$x=\frac{-(-30)\pm\sqrt{(-30)^2-4\times4\times50}}{2\times4}$$
$$=\frac{30\pm\sqrt{900-4\times4\times50}}{2\times4}$$
$$=\frac{30\pm\sqrt{100}}{8}$$
$$=\frac{30\pm10}{8}$$
$$=\frac{5}{2},5$$

We must be careful about which of these solutions to choose. Remember that, in the sphere of functions, square root means 'positive square root'.

Testing the smaller solution by substituting it in the equation:

$$\sqrt{2\times\frac{5}{2}-1}=2\times\frac{5}{2}-7$$
$$\sqrt{4}=5-7$$

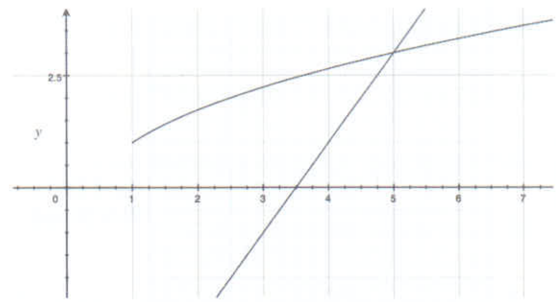
This is only true if we take the negative value of the square root and so this value does not count as a solution.

Testing 5 gives:

$$\sqrt{2\times5-1}=2\times5-7$$
$$\sqrt{9}=10-7$$

which is true.

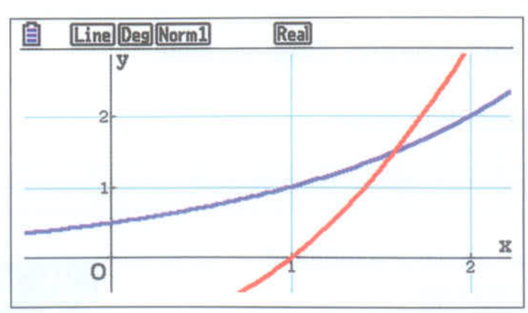
A graph will clarify the position.




Mixtures of polynomial functions and 'other' functions in equations generally do not yield easily to analytic solutions. By 'other', we mean a whole range of logarithmic, exponential, trigonometric etc. functions. As these appear frequently in applications, non-analytic solutions are an important part of mathematics. In this section, we will look at graphical methods.

**Example B.7.6**

If $f(x)=2^{x-1}, x>0$ and $g(x)=x^2-1, x>0$, find the smallest value of $x$ for which $f(x)=g(x)$.




$x\approx1.5793.$

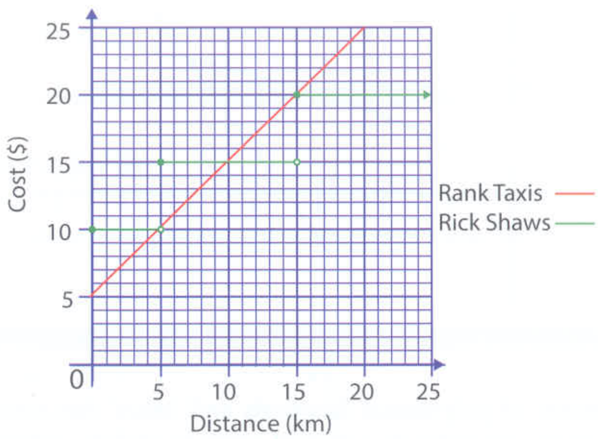
### Example B.7.7

The city of Utopia has two taxi services: Rank Taxis and Rick Shaws.

Rank Taxis charge \$5 flagfall and \$1 per kilometre.

Rick Shaws charge \$10 for all distances from zero to less than 5 km. \$15 for distances from 5 km to less than 15 km and \$20 for distances 15 km or more.

Show this graphically and find which of the two companies are cheaper at each trip distance.




The dots and rings on the green graph are significant when answering this question. It looks like there is nothing to choose between the two companies for a distance of 5 km. However there is a 'ring' on the green graph indicating that the point is excluded - so Rank Taxis are better for this distance.

The cheapest options are:

$0 \le$distance$<10$ km: Rank Taxis.

10 km, both companies charge the same.

$10<$distance$<15$ km: Rick Shaws.

15 km, both companies charge the same.

$15<$distance: Rick Shaws

Video of calculator solution of:

$$\frac{2}{x^2+1}=e^x$$

## Exercise B.7.3

1. Find the values of $x$ for which $f(x) = g(x)$.

a. $f(x)=\frac{1}{x}, x>0$
$g(x)=2x-1, x>0$

b. $f(x)=\frac{1}{x^2+1}, x>0$
$g(x)=x^2, x>0$

c. $f(x)=\frac{1}{\sqrt{x-2}}, x>2$
$g(x)=2-x, x>0$

d. $f(x)=x^3-1, x>1$
$g(x)=\sqrt{x-1}, x>1$

e. $f(x)=\log_{10}(x+1), x>0$
$g(x)=\frac{3}{x}, x>0$

f. $f(x)=2^{x-1}, x\in\mathbb{R}$
$g(x)=3^x, x\in\mathbb{R}$

g. $f(x)=(x-1)^3, x\in\mathbb{R}$
$g(x)=5-x, x\in\mathbb{R}$

h. $f(x)=\sqrt[3]{x+1}, x>0$
$g(x)=2x-5, x\in\mathbb{R}$

i. $f(x)=\frac{1}{x+\sqrt{x}}, x>0$
$g(x)=(x+1)^2, x\in\mathbb{R}$

j. $f(x)=\cos x, 0<x<\frac{\pi}{2}$
$g(x)=\sqrt{x}, x>0$

k. $f(x)=e^x, x\in\mathbb{R}$
$g(x)=x^2-3, \in\mathbb{R}$

l. $f(x)=\ln\left(\frac{1}{x+1}\right), x>0$
$g(x)=3-x, x\in\mathbb{R}$

2. The concentration of reagent A in a reaction mixture is initially 340 g/L and decreases by 10% per hour. The initial concentration of the product (B) is zero and rises by 25 g/L every hour.

   a Write functions to represent the concentrations of A and B at time $t$.

   b When will the concentrations be equal?

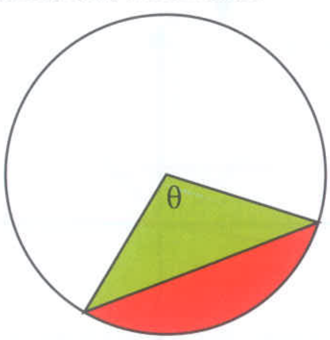
3. An investment of $10 000 is placed in a deposit account that pays 6% annual interest compounded monthly.

   a Write a function that models the amount of this investment $t$ months after it is placed.

   b When will the value of the investment reach its target of $12 500?

4. Find the value of $a$ such that if $f(x)=a^x$ and $g(x)=x^a$ then $f(3)=g(3)$.

5. The diagram shows a unit circle.




   a Find a function that gives the area of the green triangle in terms of $\theta$.

   b Find a function that gives the area of the shaded sector (red and green) in terms of $\theta$.

   c Find a function that gives the area of the red shaded segment in terms of $\theta$.

   d Write an equation that gives the fraction of the area of the circle occupied by red segment.

   e When does the segment occupy one quarter of the area of the circle?

6. Two functions are defined as follows:

   $A(x)$ is a logarithmic function of the form:

   $A(x)=a\times\ln(bx)$ with these values:

| $x$ | 1.0 | 1.5 | 2.0 | 2.5 | 3.0 |
|---|---|---|---|---|---|
| $A(x)$ | 3.140 | 4.113 | 4.804 | 5.339 | 5.777 |

   $B(x)$ is an exponential function of the form:

   $B(x)=c^x$ with these values:

| $x$ | 1.0 | 1.5 | 2.0 | 2.5 | 3.0 |
|---|---|---|---|---|---|
| $B(x)$ | 1.600 | 2.024 | 2.560 | 3.238 | 4.096 |

   a Find values of $a$ & $b$ and hence define $A(x)$.

   b Find value of $c$ and hence define $B(x)$.

   c Find the value of $x$ such that $A(x) = B(x)$

7. How long will it take for $1 000 invested at 6% annual interest compounded monthly to exceed $1 100 invested at 6% annual simple interest.

8. If a fixed mass of gas is kept at a constant temperature, its pressure ($p$) and volume ($v$) are relared by Boyle's Law: $pv$ = constant.

   If, however, a fixed mass of gas is suddenly compressed, the temperature rises and the pressure is, as a result larger than it would have been at constant temperature.

   This sort of sudden compression is commonplace in engines and other sorts of machinery.

   In these cases, Boyle's Law is often amended to $pv^\gamma$ = constant. $\gamma$ is also a constant.

   Here are some experimental measurements

| $p$ | 0.7 | 1.5 | 1.9 | 2.3 | 3.6 |
|---|---|---|---|---|---|
| $v$ | 3.60 | 1.68 | 1.33 | 1.10 | 0.70 |

   Find the value of $\gamma$.

Answers


The idea of a function is very general. In school mathematics texts, it is usually confined to rules such as:

$$f(x) = x^2 - x + 2$$

as we have been discussing.

In these, the domains and ranges are real numbers. This need not be the case.

Every human being has a blood type (O, A, AB etc.). The 'function' that has as it domain {all the people on Earth} and which returns that person's blood type is a well defined many to one function that can be very important to those who need a transfusion. Yet it is not expressible as an algebraic rule.

As a second example, every credit card in the world is identified by a number (often with sixteen digits). This number is linked to the account that the bank keeps of the transactions. The rule connecting these two attributes is, in a very real sense, a function. It needs also to be a one to one function if the banking system is not to collapse in chaos. However, it is a function that is not expressed as an algebraic rule. Rather, it is a list of pairs (card number, account number) stored in a computer.

The connection between a credit card and the associated PIN number is more subtle and students are encouraged to read about it. An internet search using ' Public key cryptography' will get you started.

The notion of a function is even wider than this. In the last section of this course, you will be introduced to Calculus. This is a branch of mathematics which transforms functions. You can think of it as functions that have functions as their domain.

There is also a class of 'functions' that use shapes and objects as the elements of their domain.

These take a shape and transform it into another shape using a single mathematical rule (i.e. function).

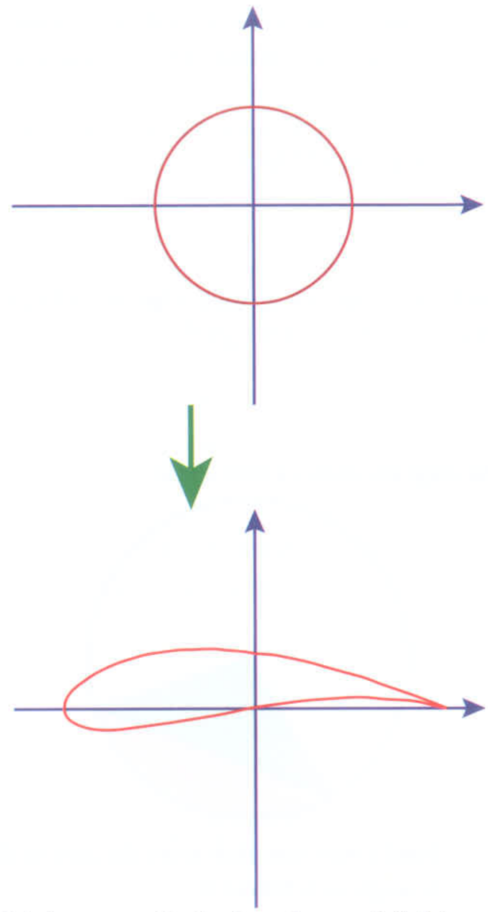
## Joukowsky transformation

As an example, we will look briefly at a 'function' known as the Joukowsky transformation.

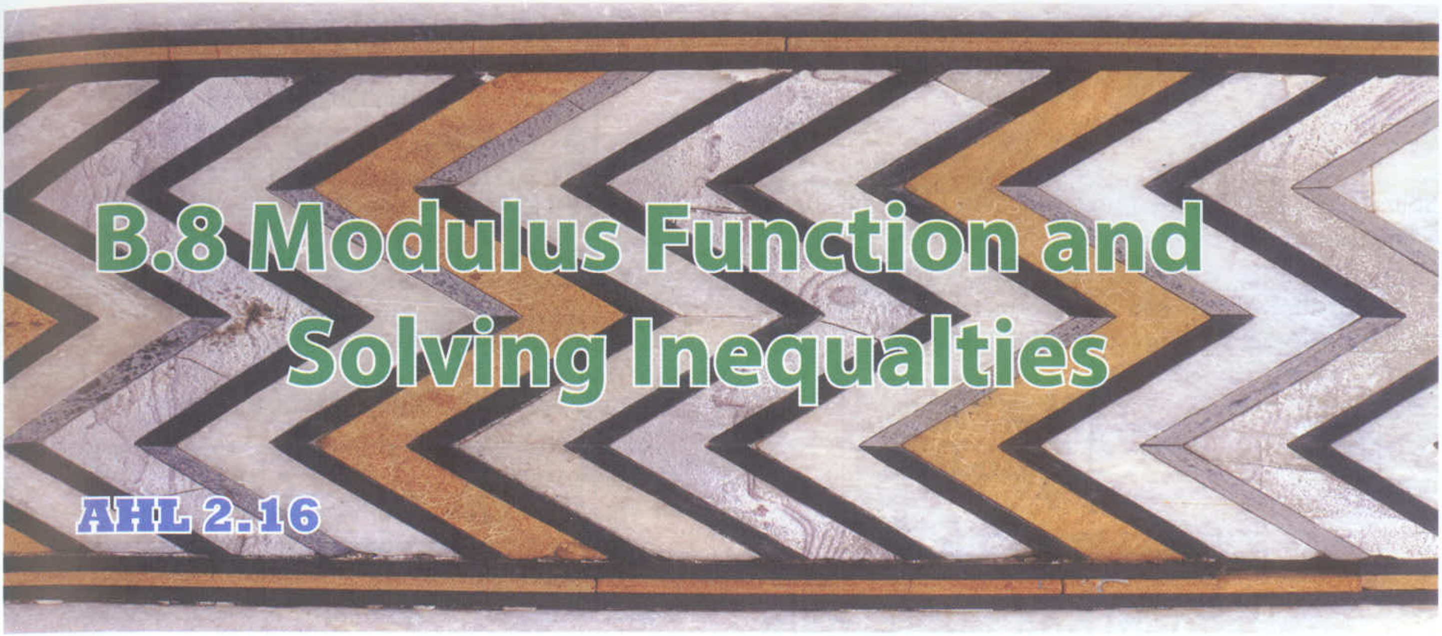
This has the rule $f(z) = z + \frac{1}{z}$.

The variable $z$ is taken from the set of complex numbers. Since functions of complex numbers do not form part of this course, we will not go into details but concentrate on the broad picture.

If we take a unit circle as the element of the domain, the Joukowsky transformation changes it in a way depicted below:



The result is known as the Joukowsky aerofoil. It turns out that it is not the best design for an aeroplane wing. However, the ability to use a mathematical process that can take a simple shape such as a circle and convert it to a much more complex and useful shape is valuable. It can, for example, be used to send precise instruction to robotic milling machines.



# B.8 Modulus Function and Solving Inequalties

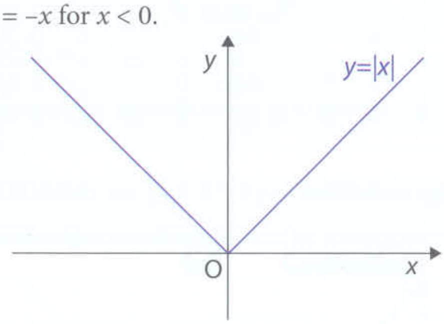
AHL 2.16

## The Absolute Value Function

The absolute value function is defined as

$$f(x)=\begin{cases} x \text{ if } x\geq 0 \\ -x \text{ if } x<0 \end{cases}$$

That is, sketch the graph of $y = x$ for $x \geq 0$, and then sketch the graph of $y = -x$ for $x < 0$.




Similarly, the function $f(x) = |ax + b|$ , represents the absolute value of the linear function $y = ax + b$.

**Example B.8.1**

Sketch the graphs of:

a $y = |x - 2|$

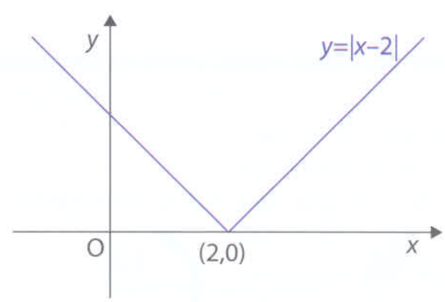
b $y = |x| + 1$

c $y = |2x + 1|$

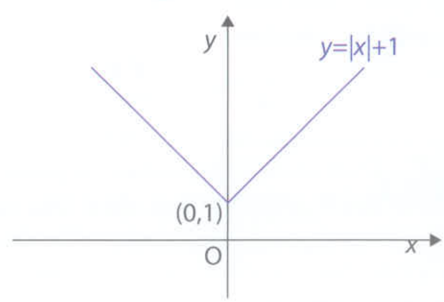
Parts a and b are best done by considering the functions as translations of the basic absolute value function. That is, the graph of $y = |x - 2|$ is the graph of $y = |x|$ translated two units to the right.

The graph of $y = |x| + 1$ is the graph of $y = |x|$ translated one unit vertically up. So, we have:

a




b




c $y = |2x + 1|$

This function can be seen in two parts:

If $2x + 1 \geq 0$ or $x \geq -{}^{1}/_{2}$, $y = 2x + 1$.

If $2x + 1 < 0$ or $x < -{}^{1}/_{2}$, $y = -(2x + 1) = -2x - 1$.

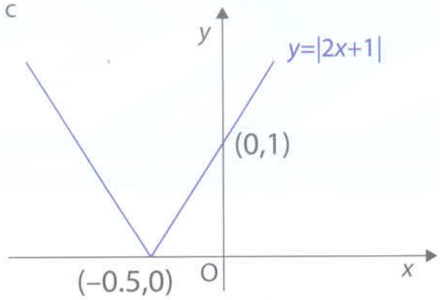
A transformation would also work but you must factorize first to get:

$$y=|2x+1|=\left|2\left(x+\frac{1}{2}\right)\right|$$

Shift the graph of $y=|x|$ horizontally left by ½ and stretch along the $y$-axis by factor = 2.

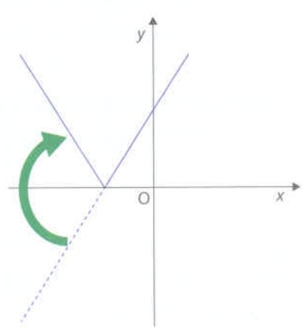
If you multiply/divide and add/subtract from a variable $x$ then you must factorize first to get the right graph.

c




Video discussion

Notice that in fact, all we have done in part c is to sketch the graph of $y = 2x + 1$, and then reflect (about the $x$-axis) any part of the graph that was drawn below the $x$-axis. We can also make use of graphic calculators to sketch graphs of absolute value functions:




## Example B.8.2

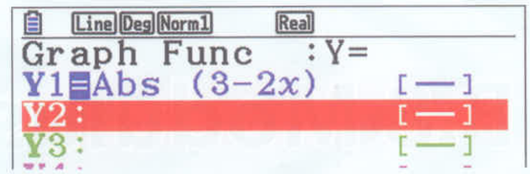
Find the range of the following functions:

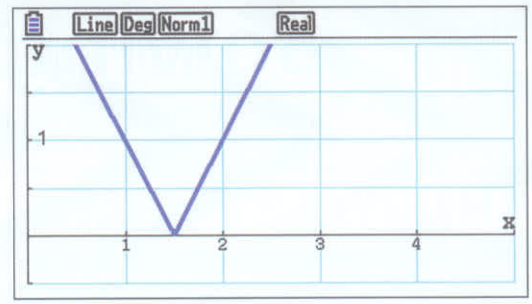
a $f(x)=|3-2x|, x\in\mathbb{R}$

b $f(x)=|x+1|+|x-1|, x\in\mathbb{R}$

c $f(x)=|x-4|-2, x\in\mathbb{R}$

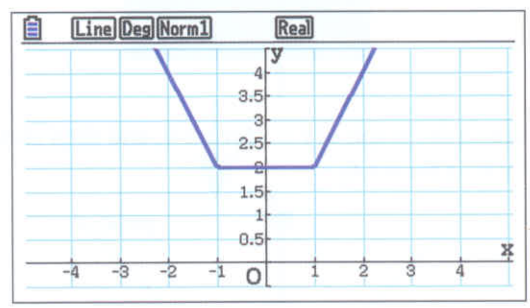
a Use the GRAPH module. The abs( option is usually found under a MATH menu. CASIO have this under OPTN, F5 NUMERIC, F1 Abs. After 'pasting' the Abs command, enter the equation as shown on the screen. Then select the equation (EXE) and F6-DRAW.



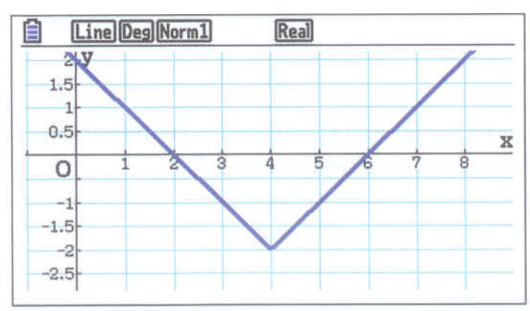

From the given graph, the range is defined as $\{y: y\geq 0\}$.

b As before, we enter the required options and obtain the following:




Range is defined as $\{y: y\geq 2\}$ or $[2.\infty)$.

c



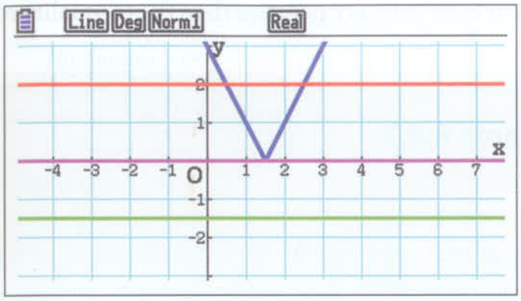

Range is defined as $\{y: y\geq -2\}$ or $[-2.\infty)$.

## Example B.8.3

Find the number of solutions of each of these equations:

a $|3-2x|=2$ b $|3-2x|=-1.5$ c $|3-2x|=0$

Using the graph from Example B.8.2 a.



a $|3-2x|=2$ $y=2$ is the red horizontal line. It intersects with the blue graph in 2 places. There are 2 solutions.

b $|3-2x|=-1.5$ , $y=-1.5$ is the green horizontal line. It does not intersect the blue graph. There are no solutions.

c $|3-2x|=0$ $y=0$ is the magenta horizontal line. It intersects with the blue graph in 1 place. There is 1 solution.

**Example B.8.4**

Find the value $p \in \mathbb{R}$ such that the equation

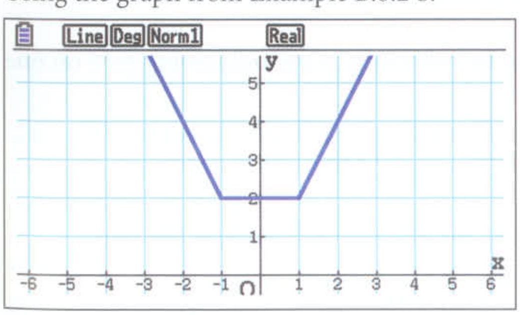
$|x+1|+|x-1|=p$ has:

a no solution

b infinitely many solutions

c two solutions.

Using the graph from Example B.8.2 b.



a no solution: $p < 2$.

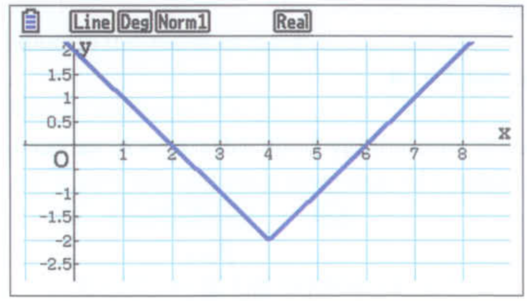
b infinitely many solutions: $p = 2$.

c two solutions: $p > 2$.

**Example B.8.5**

Find the value $k \in \mathbb{R}$ such that the equation $|x-4|-2=k$ has exactly one solution.

Using the graph from Example B.8.2 c, $k = -2$.



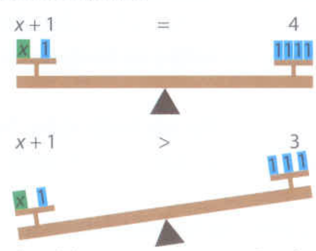
## Inequalities and Inequations

The terms inequality and inequation are similar. Inequation usually refers to a statement such as $x + 2 \geq 7$ where one quantity is bigger than or equal to another. Inequality usually refers to a statement such as $x + 2 > 7$ where one quantity is strictly bigger than another.

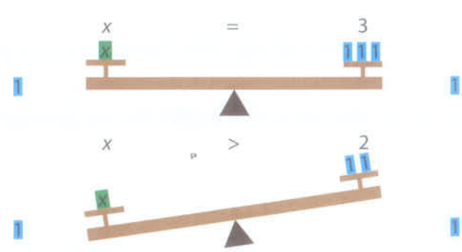
Solving an inequality means that we find ALL possible values for which the inequality holds, all values which satisfy the inequality.

Inequalities very often have infinitely many solutions. Therefore solutions cannot be checked by substitution.

If equations are like a child's seesaw in balance, inequations are like unbalanced seesaws.




Equations are solved by maintaining the balance In the case of our example, subtracting 1 from both sides simplifies the problem and leads to a solution. Much as subtracting one from both sides of a balanced seesaw maintains the balance, the same process leaves an unbalanced seesaw unbalanced.



For this reason, solving inequations is similar to solving equations. With one major exception.

Major issues you have to pay attention to when solving inequalities:

Multiplication/division by a negative number reverses the inequality.

It is true that $7 > -4$.

However, if we multiply both sides by $-2$, we get $-14 > 8$ which is **false**. The statement does become true if we reverse the sign:

$$7 > -4 \text{ (multiply by } -2) \Rightarrow -14 < 8$$

The same is true of division by a negative number.

**Example B.8.6**

Solve: a $2x < 3x + 4$

b $5x - 6 > -4x + 3$

a $2x < 3x + 4$ Subtract $3x$ from both sides

$-x < 4$ Mutiply by $-1$ and reverse sign

$x > -4$

b $5x - 6 > -4x + 3$ Add $4x$ to both sides

$9x - 6 > 3$ Add 6 to both sides

$9x > 9$ Divide both sides by 9.

$x > 1$

No sign reversal is necessary.

## Reciprocals

Care is needed with inequations involving reciprocals.

It is true that $3 > 2$. But what if we take the reciprocals of both sides?

$\frac{1}{3} > \frac{1}{2}$ is **false**.

$-3 < -2$ is also true, but $-\frac{1}{3} < -\frac{1}{2}$ is **false**.

If taking reciprocals, when both sides of the inequality are positive or both sides are negative then the inequality reverses.

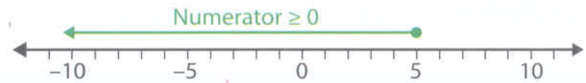
**Example B.8.7**

Solve: a $\frac{5-x}{2x+1} \geq 0$ b $\frac{1}{x-4} > 4$

c $\frac{1}{x} > \frac{1}{2x-1}$

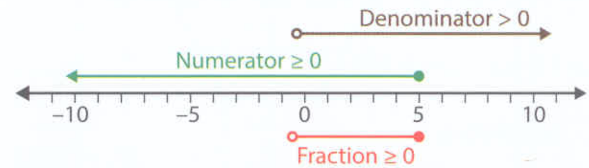
a We are going to represent on a number line how the numerator and denominator change sign as $x$ varies.

Let us check when the numerator is positive ($5 - x \geq 0$)




We do the same with the denominator:

If $2x + 1 > 0$ then $2x > -1$ and $x > -½$.




The whole fraction is positive when both the numerator and denominator are positive (red interval).

The case when both numerator and denominator are negative ($x < -½$ and $x \geq 5$) has no solutions.

The final solution is $-½ < x \leq 5$.

b We begin by collecting terms and leaving zero on one side:

$$\frac{1}{x-4} > 4 \Rightarrow \frac{1}{x-4} - 4 > 0$$

Next, make the left hand side a single fraction:

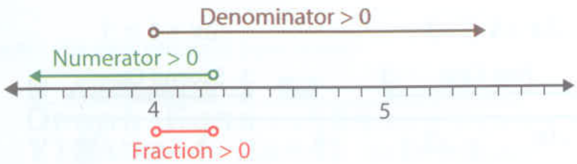
$$\frac{1}{x-4} - \frac{4(x-4)}{x-4} > 0$$

$$\frac{1-4(x-4)}{x-4} > 0$$

$$\frac{1-4x+16}{x-4} > 0$$

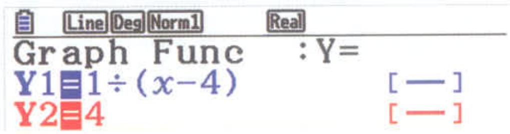
$$\frac{17-4x}{x-4} > 0$$

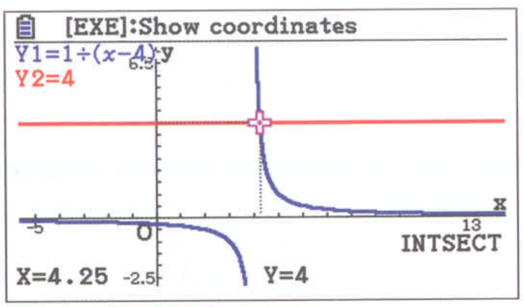
Using a number line:




Solution $4 < x < 4.25$

It is a good idea to back up your algebra with a graphical approach. Enter both sides of the inequation as separate functions:




c $\frac{1}{x} > \frac{1}{2x-1}$

Collecting terms as before:

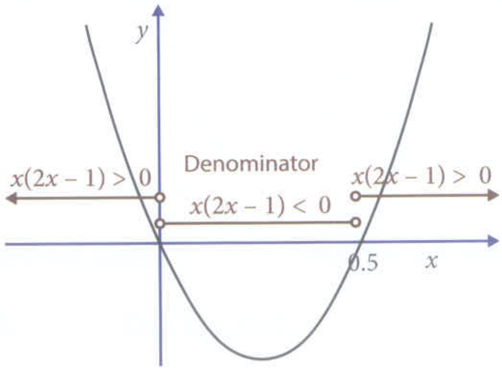
$$0 > \frac{1}{2x-1} - \frac{1}{x}$$

$$0 > \frac{x}{x(2x-1)} - \frac{2x-1}{x(2x-1)}$$

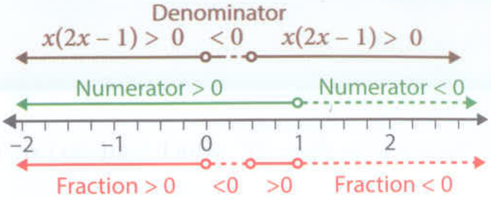
$$0 > \frac{x-(2x-1)}{x(2x-1)}$$

$$0 > \frac{1-x}{x(2x-1)}$$

We are going to see how the denominator changes sign by looking at the graph of $y = x(2x - 1)$.

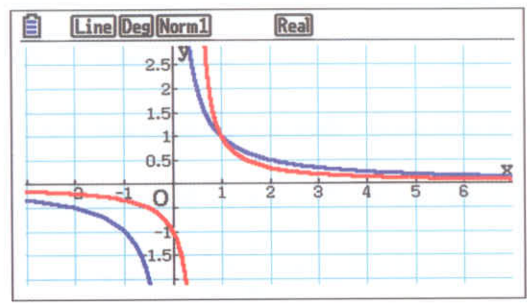



Combining this result with the sign changes of the numerator $(1 - x)$.




Solution $0 < x < ½$ or $x > 1$

This is confirmed by the graph. We are looking for intervals in which the blue graph is above the red.




Note: If the sides of the inequality have different signs then the inequality sign remains the same when taking the reciprocals of both sides.

Example: If $-2 < 3$ then $-½ < {}^1/_3$

## Squares and square roots

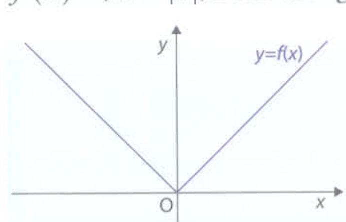
On squaring both sides of an inequality: The inequality sign does not change when both sides are positive or both sides are negative.

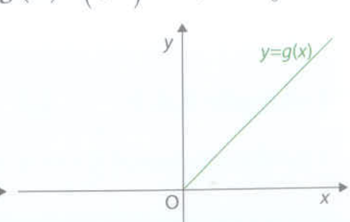
Example: $3 > 2$ then $3^2 > 2^2$ but $-2 < -3$ then $(-2)^2 > (-3)^2$.

Taking the root of both sides of an inequality: You have to make sure that both sides of the inequality are non-negative before taking the roots of both sides. The inequality sign does not change.

Remember the difference between the two functions:

$$f(x) = \sqrt{x^2} = |x|, x \in \mathbb{R} \text{ and } g(x) = \left(\sqrt{x}\right)^2 = x, x \in \mathbb{R}_0^+.$$



**Example B.8.8**

Solve: $x^2 > (x-3)^2$

$x^2 > (x-3)^2$ then $\sqrt{x^2} > \sqrt{(x-3)^2}$ which leads to $|x| > |x-3|$.

This is true for $x > 1.5$.

## Summary

If $a > b$ and $c \in \mathbb{R}$ then $a + c > b + c$ and $a - c > b - c$

If $a > b$ and $c \in \mathbb{R}^+$ then $a \times c > b \times c$ and $\frac{a}{c} > \frac{b}{c}$

If $a > b$ and $c \in \mathbb{R}^-$ then $a \times c < b \times c$ and $\frac{a}{c} < \frac{b}{c}$

If $a > b$ and $a, b \in \mathbb{R}^+$ then $\frac{1}{a} < \frac{1}{b}$

If $a > b$ and $a, b \in \mathbb{R}^-$ then $\frac{1}{a} > \frac{1}{b}$

If $a > b$ and $a, b \in \mathbb{R}^+$ then $a^2 > b^2$

If $a > b$ and $a, b \in \mathbb{R}^-$ then $a^2 < b^2$

If $a > b$ and $a, b \in \mathbb{R}^+$ then $\sqrt{a} > \sqrt{b}$

# Modulus Inequalities

The simplest types of inequalities are shown in the table.

| | $\|A\|<0$ | $\|A\|\le 0$ | $\|A\|>0$ | $\|A\|\ge 0$ |
|---|---|---|---|---|
| solution | no solution | $A = 0$ | $A \ne 0$ | $A \in \mathbb{R}$ |

| | $\|A\|<-2$ | $\|A\|\le -2$ | $\|A\|>-2$ | $\|A\|\ge -2$ |
|---|---|---|---|---|
| solution | no solution | no solution | $A \in \mathbb{R}$ | $A \in \mathbb{R}$ |

| | $\|A\|<3$ | $\|A\|\le 3$ | $\|A\|>3$ | $\|A\|\ge 3$ |
|---|---|---|---|---|
| solution | $-3<A<3$ | $-3\le A\le 3$ | $3<A$ <br> $A<-3$ | $3\le A$ <br> $A\le -3$ |

**Example B.8.9**

Find the solution of $|-2x+4| \ge 3$, where $x$ is a real number.

There are two choices:

$-2x + 4 \le -3$ $\qquad$ $-2x + 4 \ge 3$


| | |
|---|---|
| $-2x + 4 \le -3$ | $-2x + 4 \ge 3$ |
| $\Rightarrow -2x \le -7$ | $\Rightarrow -2x \ge -1$ |
| $\Rightarrow x \ge {}^7/_2$ | $\Rightarrow x \le ½$ |

Solution: $(-\infty, ½] \cup [{}^7/_2, \infty)$

**Alternative solution:**

Squaring both sides of the inequality will not change the inequality sign since both sides are non-negative.

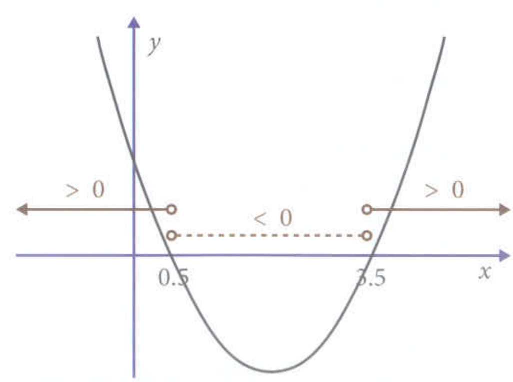
$$|-2x+4|^2 \ge 3^2$$

Using that we can change the modulus inequality for a quadratic inequality.

$$(-2x+4)^2 \ge 9$$
$$4x^2 - 16x + 7 \ge 0$$

This is a 'vertex down' parabola. Solving for the $x$-intercepts:

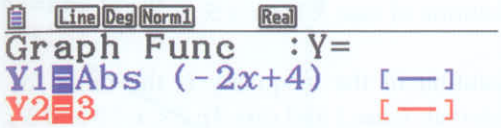
$$x = \frac{-(-16) \pm \sqrt{(-16)^2 - 4 \times 4 \times 7}}{2 \times 4}$$
$$= \frac{16 \pm \sqrt{144}}{8}$$
$$= \frac{16 \pm 12}{8}$$
$$= \frac{1}{2}, \frac{7}{2}$$



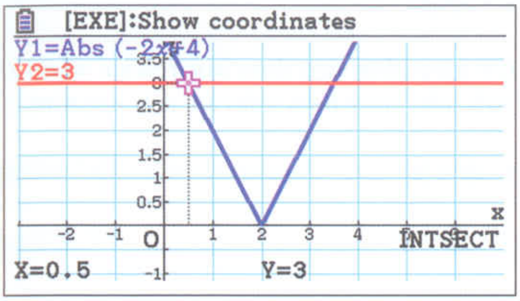

Solution: $(-\infty, ½] \cup [{}^7/_2, \infty)$

A third method is graphical.

First enter the two relevant functions:




Looking at the sense of the inequation (Y1≥Y2), we will be looking for intervals for which the blue graph is above the red.




It may be necessary to locate intersection points. In this case, (0.5,3) is highlighted.

Solution: $(-\infty, \frac{1}{2}] \cup [\frac{7}{2},\infty)$

**Example B.8.10**

Solve:

a $|5 - 2x| < 1$ b $||2x - 1| - 3| > 2$

c $||3x - 5| + 4| \le 7$

d $|x^2 - 3x + 2| = 3x - x^2 - 2$

e $|x^2 - 2x - 3| < 3x - 3$

f $|1 - x| < |x|$

a $-1 < 5 - 2x < 1$

$-6 < -2x < -4$

$3 > x > 2$ satisfy the inequality.

b $||2x - 1| - 3| > 2$

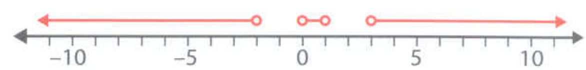
There are two cases:

Case A: $|2x - 1| - 3 > 2$

Case B: $|2x - 1| - 3 < -2$


| | |
|---|---|
| $\|2x - 1\| - 3 < -2$ | $\|2x - 1\| - 3 > 2$ |
| $\|2x - 1\| < 1$ | $\|2x - 1\| > 5$ |
| $-1 < 2x - 1 < 1$ | There are two choices: |

| | | |
|---|---|---|
| $0 < 2x < 2$ | $2x - 1 < -5$ | $2x - 1 > 5$ |
| $0 < x < 1$ | $2x < -4$ | $2x > 6$ |
| | $x < -2$ | $x > 3$ |




Solution: $x < -2$, $0 < x < 1$, $x > 3$.

c $||3x - 5| + 4| \le 7$

$-7 \le |3x - 5| + 4 \le 7$

$-11 \le |3x - 5| \le 3$

$-11 \le |3x - 5|$ is true for all values of $x$.

$|3x - 5| \le 3$ is satisfied when $-3 \le 3x - 5 \le 3$

$2 \le 3x \le 8$

$\frac{2}{3} \le x \le \frac{8}{3}$

Therefore the solution of the inequality is $\frac{2}{3} \le x \le \frac{8}{3}$

d $|x^2 - 3x + 2| = 3x - x^2 - 2$

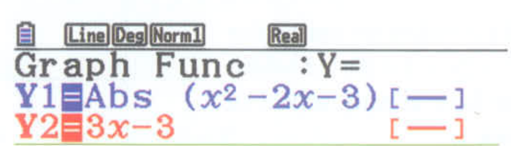
$|x^2 - 3x + 2| = -1(x^2 - 3x + 2)$

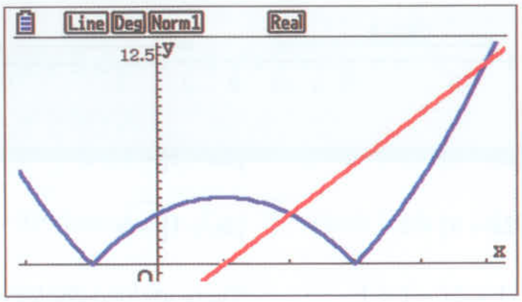
holds when $x^2 - 3x + 2 \le 0$.

$(x - 1)(x - 2) \le 0$

Solution is: $1 \le x \le 2$

e $|x^2 - 2x - 3| < 3x - 3$ Before tackling this algebraically, we will look at it graphically:





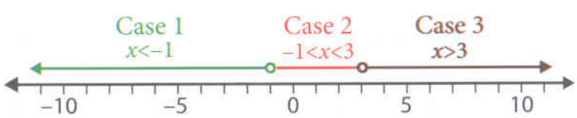

We are looking for the interval(s) in which the red graph is above the blue. G-Solv gives the intersection points as (2,3) and (5,12). This suggests that the solution is the interval $2 < x < 5$.

Algebraic solution:

$x^2 - 2x - 3 = 0$ gives $x_1 = -1$ and $x_2 = 3$.

$$|x^2 - 2x - 3| = \begin{cases} x^2 - 2x - 3, x \le -1 \text{ or } 3 \le x \\ -x^2 + 2x + 3, -1 < x < 3 \end{cases}$$

We must consider three cases:



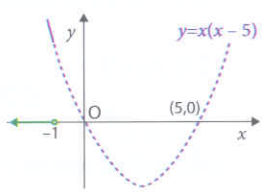

Case 1: $x \le -1$

If $x \le -1$ then we must solve $x^2 - 2x - 3 < 3x - 3$

$x^2 - 5x < 0$

$x(x - 5) < 0$

$0 < x < 5$

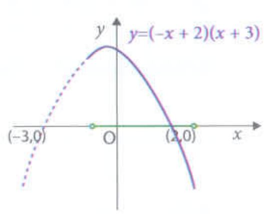



None of these values fall into the interval of $x \le -1$ therefore this case does not provide us with a solution.

Case 2: $-1 < x < 3$

If $-1 < x < 3$ then we must solve $-x^2 + 2x + 3 < 3x - 3$

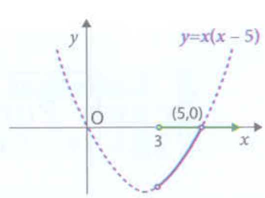
$-x^2 - x + 6 < 0$




$x < -3$, $2 < x$ We must choose those values which fall into $-1 < x < 3$. These are: $2 < x < 3$.

Case 3: $x > 3$

If $x > 3$ then we must solve $x^2 - 2x - 3 < 3x - 3$ which we have already solved in case 1.




This time some of the $0 < x < 5$ values fall in the $x > 3$ interval.

Solution of case 3: $3 < x < 5$

Solution of the inequality is the union of the two intervals (case 2 and case 3): $2 < x < 3$ and $3 < x < 5$.

Solution : $2 < x < 5$.

f $|1 - x| < |x|$

The inequality sign does not reverse when squaring both sides since both sides are non-negative.

$|1-x|^2 < |x|^2$

$(1 - x)^2 < x^2$

$x^2 - 2x + 1 < x^2$

$-2x + 1 < 0$

$-2x < -1$

$2x > 1$

$x > ½$

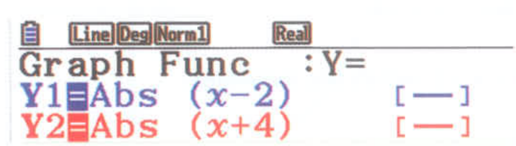
## Graphical Approach

We have already suggested that a graphical approach coupled with the use of technology can be very helpful with the more complex of these problems.

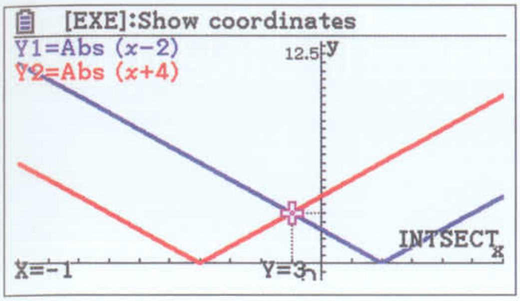
**Example B.8.11**

Solve $|x - 2| \le |x + 4|$

Enter the functions:



We need to look for the red graph being above the blue.




The solution is $x \geq -1$.

## Exercise B.8.1

1. On separate sets of axes, sketch the graphs of the following functions for $x \in [-5,5]$.

a $f(x)=|2x|$ b $f(x)=|x|+2$

c $f(x)=|4x-2|$ d $f(x)=|1-x|$

e $f(x)=\left|-1-\frac{1}{2}x\right|$ f $f(x)=\left|\frac{1-x}{2}\right|$

2 On separate sets of axes, sketch the graphs of the following functions for $x \in [-5,5]$.

a $f(x)=|x^2-9|$

b $f(x)=|2x-x^2|$

c $f(x)=|(x-1)(2+x)|$

d $f(x)=|x||x+1|$

e $f(x)=|8-x^3|$

f $f(x)=5-|x^3|$

3 On separate sets of axes, sketch the graphs and determine the ranges of each of the following functions.

a $f(x)=|x+1|+|x-1|$

b $f(x)=|x+2|+|x-2|$

c $f(x)=x+|x|$

d $f(x)=x-|x|$

e $f(x)=|x+2|-|x-2|$

4 On separate sets of axes, sketch the graphs of the following functions.

a $f(x)=x|x|$ b $f(x)=\frac{x}{|x|}, x \neq 0$

c $f(x)=\left|\frac{1}{x}+1\right|, x \neq 0$

d $f(x)=\left|\frac{1}{x}-1\right|, x \neq 0$

5. Solve the following inequalities for $x \in \mathbb{R}$.

a $3x - 5 \geq 5x - 9$ b $x - 9 \geq 4x - 2$

c $3 - 8x < 2x + 1$ d $\frac{2}{x} > x+1, x \neq 0$

e $x^2 < 6 - x$ f $x^2 + x < 2$

g $x^2 - 3x > 4$ h $x^2 + 2x + 5 > -4$

i $x^2 + 2x + 5 < -4$ j $x^2 + x > 1$

k $x^2 \leq 3 - x$ l $x^2 \leq 2x + 5$

6. Solve the following inequalities for $x \in \mathbb{R}$.

a $|x| \leq 4$ b $|x - 1| \leq 4$

c $|x + 1| \geq x$ d $|x - 1| \geq x$

e $|x + 1| \geq x^2$ f $|x + 1| \geq |x - 1|$

g $|2x + 1| \geq |x - 1|$ h $|x + 1| \geq |x - 3|$

i $|2x - 3| \geq |4x - 3|$ j $|1 - 2x| \geq |x - 3|$

7. Solve the following inequalities for $x \in \mathbb{R}$.

a $|x - 1| \geq |x^2|$ b $|x+1| \geq \left|\frac{1}{x}\right|$

c $|x+1| \geq \left|\frac{1}{x-3}\right|$

d $|x - 1| \geq |x + 1| + |x - 3|$

e $|x - 1| \geq |x + 1| - |x - 3|$

f $|x - 1| \geq |x + 1| - 2|x - 3|$

g $|7x - 1| \geq |3x + 1| + |5x - 3|$

h $|x^2 - 9| = 9 - x^2$


1. Show that $a^3+b^3 \geq a \times b(a+b)$ where $0 < a, b$.

2. Show that $a^2+b^2+c^2 \geq ab+bc+ca$ where $a,b,c \in \mathbb{R}$

3. Show that $a^2+b^2+c^2+3 \geq 2(a+b+c)$ where $a,b,c \in \mathbb{R}$.

4. Show that $(a+b)(b+c)(a+c) \geq 8abc$ where $a,b,c \in \mathbb{R}$

5. Show that $\frac{2ab}{a+b} \leq \frac{a+b}{2}$ where $a,b \in \mathbb{R}^+$.

6. Show that $\frac{a}{b^2}+\frac{b}{a^2} \geq \frac{1}{a}+\frac{1}{b}$ where $0 < a, b$.

7. Show that $\left(\frac{a+b}{2}\right)^2 \leq \frac{a^3+b^3}{2}$ where $a,b \in \mathbb{R}^+$.

8. Prove that the sum of a positive number and its reciprocal is always greater than or equal to 2.

9. Show that $\frac{2ab}{a+b} \leq \frac{a+b}{2}, a,b \in \mathbb{R}^+$.

10. Show that $\frac{a+2}{a}+\frac{a+2}{2} \geq 4, a,b \in \mathbb{R}^+$.

11. Find the smaller of $3^{185}$ and $6^{85}$.

12. Given $a \geq b$, $x \leq y$, prove that $\frac{ax+by}{y} \leq \frac{a+b}{2} \times \frac{x+y}{2}$

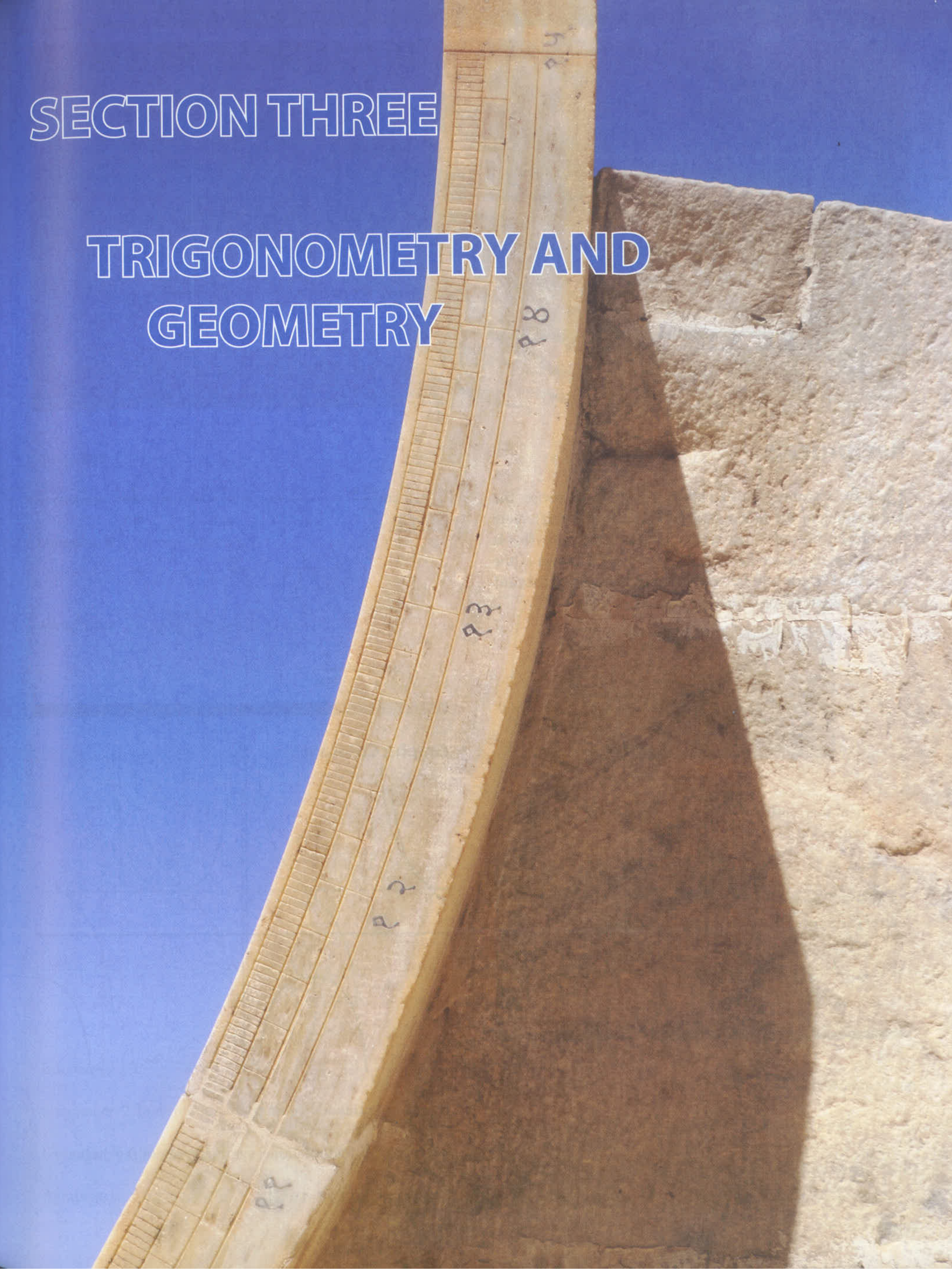
13. For how many $n \in \mathbb{Z}^+$ is $(n+1)^n > n^{n+1}$.

14. Prove that for all $x \in \mathbb{R}, \frac{x^3-1}{3} \leq \frac{x^4-1}{4}$.

Answers

# SECTION THREE

## TRIGONOMETRY AND GEOMETRY



# C.8 Reciprocal and Inverse Trigonometric Functions

AHL 3.9

## Reciprocals

Before beginning this chapter, you should revise the meaning of the radian (Chater C4 of the SL text). Note the introduction of a new trigonometric ratio, cotθ. This is one of a set of three other trigonometric ratios known as the **reciprocal trigonometric ratios**, namely **cosecant**, **secant** and **cotangent** ratios. These are defined as:

$$\text{cosecant ratio: } \operatorname{cosec}\theta = \frac{1}{\sin\theta}, \sin\theta \neq 0$$

$$\text{secant ratio: } \sec\theta = \frac{1}{\cos\theta}, \cos\theta \neq 0$$

$$\text{cotangent ratio: } \cot\theta = \frac{1}{\tan\theta}, \tan\theta \neq 0$$

Note then, that $\cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}, \sin\theta \neq 0$ and cosec is often written 'csc'.

**Example C.8.1**

Find the exact values of:

a $\sec 45°$ b $\operatorname{cosec} 150°$

c $\cot\frac{11\pi}{6}$ d $\sec 0$

a $\sec 45° = \frac{1}{\cos 45°} = \frac{1}{\left(\frac{1}{\sqrt{2}}\right)} = \sqrt{2}$

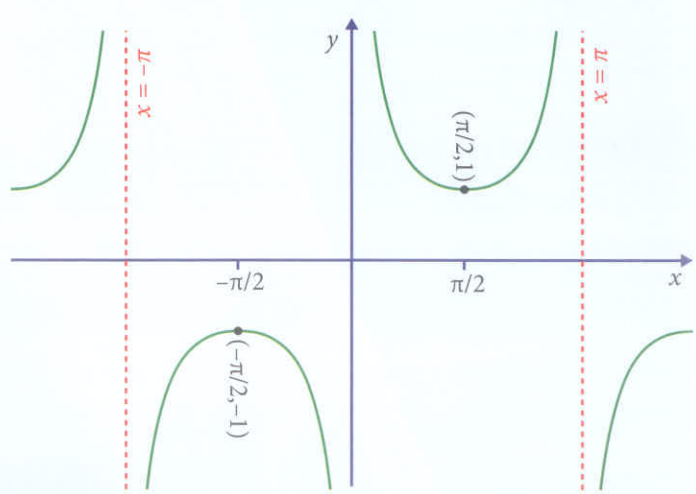
b $\operatorname{cosec} 150° = \frac{1}{\sin 150°} = \frac{1}{\sin 30°} = \frac{1}{\left(\frac{1}{2}\right)} = 2$

c $\cot\frac{11\pi}{6} = \frac{1}{\tan\left(\frac{11\pi}{6}\right)} = \frac{1}{\tan\left(-\frac{\pi}{6}\right)} = \frac{1}{-\tan\frac{\pi}{6}}$

$= \frac{1}{-\left(\frac{1}{\sqrt{3}}\right)} = -\sqrt{3}$

d $\sec 0 = \frac{1}{\cos 0} = \frac{1}{1} = 1$

### Graphs: Cosecant



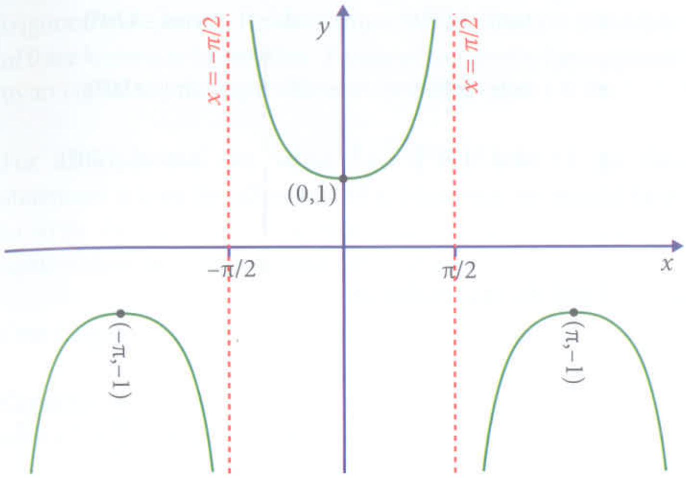

**Intercepts:** none

**Symmetry:** 2-fold rotational symmetry about the origin.

**Domain:** $\mathbb{R} / \{\pm n\pi, n \in \mathbb{Q}\}$

**Aymptotes:** $x = \pm n\pi, n \in \mathbb{Q}$

## Graphs: Secant



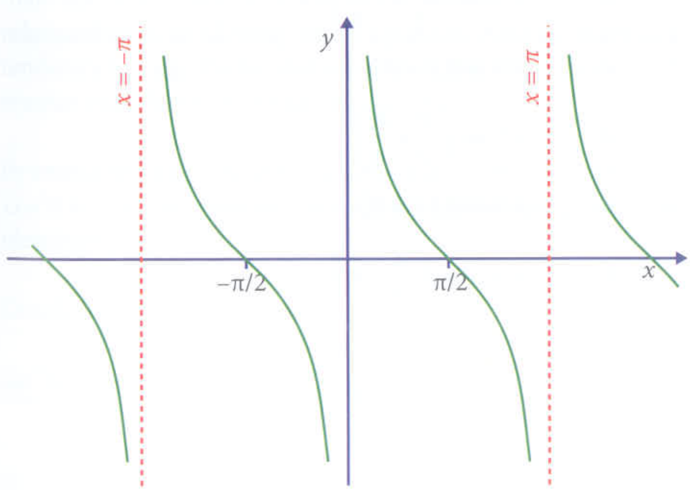

**Intercepts:** (0,1)

**Symmetry:** reflection about the $y$-axis.

**Domain:** $\mathbb{R} / \left\{ \pm \frac{2n+1}{2}\pi, n \in \mathbb{Q} \right\}$

**Aymptotes:** $x = \pm \frac{2n+1}{2}\pi, n \in \mathbb{Q}$

## Graphs: Cotangent




**Intercepts:** $\left( \pm \frac{2n+1}{2}\pi, 0 \right), n \in \mathbb{Q}$

**Symmetry:** 2-fold rotational symmetry about the origin.

**Domain:** $\mathbb{R} / \{ \pm n\pi, n \in \mathbb{Q} \}$

**Aymptotes:** $x = \pm n\pi, n \in \mathbb{Q}$

All the techniques that we have covered in relation to other graphs relate to the graphs of reciprocal trigonometric functions.

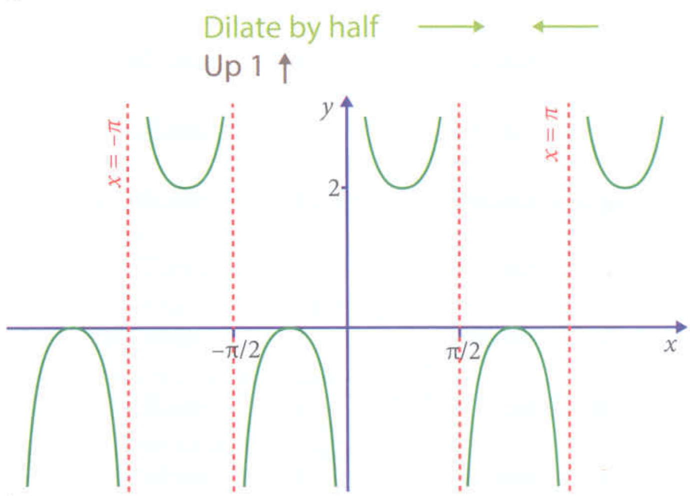
### Example C.8.2

Sketch the graphs of these functions:

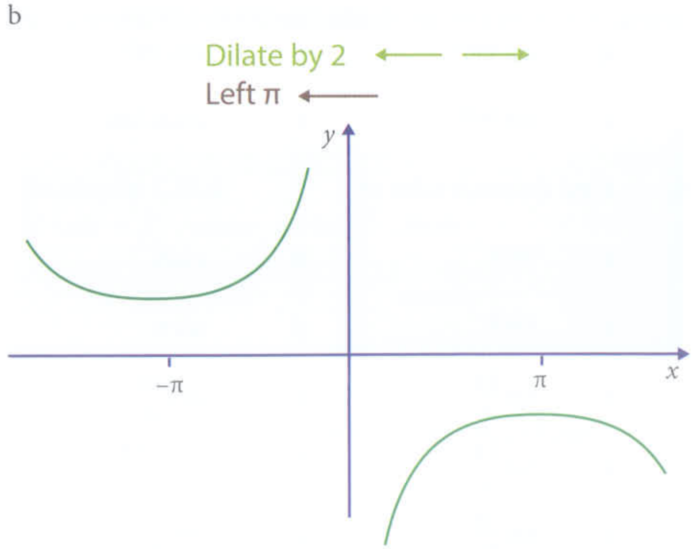
a $y = \text{cosec}(2x) + 1$ b $y = \sec\left( \frac{x+\pi}{2} \right)$

What are the x intercepts of $y = \text{cotan}\left( x - \frac{\pi}{4} \right)$

a




b




For $y = \text{cotan}\left( x - \frac{\pi}{4} \right)$, the intercepts of the basic graph are translated $\frac{\pi}{4}$ to the right: $\left( \pm \frac{2n+1}{2}\pi + \frac{\pi}{4}, 0 \right), n \in \mathbb{Q}$.

## Exercise C.8.1

1. Convert the following angles to degrees.

| | | | |
|---|---|---|---|
| a | $\frac{2\pi}{3}$ | b | $\frac{3\pi}{5}$ |
| c | $\frac{12\pi}{10}$ | d | $\frac{5\pi}{18}$ |

2. Convert the following angles to radians.

| | | | |
|---|---|---|---|
| a | 180° | b | 270° |
| c | 140° | d | 320° |

3. Find the exact value of:

| | | | |
|---|---|---|---|
| a | $\sin 120°$ | b | $\cos 120°$ |
| c | $\tan 120°$ | d | $\sec 120°$ |
| e | $\sin 210°$ | f | $\cos 210°$ |
| g | $\tan 210°$ | h | $\cot 210°$ |
| i | $\sin 225°$ | j | $\cos 225°$ |
| k | $\tan 225°$ | l | $\text{cosec } 225°$ |
| m | $\sin 315°$ | n | $\cos 315°$ |
| o | $\tan 315°$ | p | $\sec 315°$ |
| q | $\sin 360°$ | r | $\cos 360°$ |
| s | $\tan 360°$ | t | $\text{cosec} 360°$ |

4. Find the exact value of:

| | | | |
|---|---|---|---|
| a | $\sin\pi$ | b | $\cos\pi$ |
| c | $\tan\pi$ | d | $\sec\pi$ |
| e | $\sin\frac{3\pi}{4}$ | f | $\cos\frac{3\pi}{4}$ |
| g | $\tan\frac{3\pi}{4}$ | h | $\text{cosec}\frac{3\pi}{4}$ |
| i | $\sin\frac{7\pi}{6}$ | j | $\cos\frac{7\pi}{6}$ |
| k | $\tan\frac{7\pi}{6}$ | l | $\cot\frac{7\pi}{6}$ |
| m | $\sin\frac{5\pi}{3}$ | n | $\cos\frac{5\pi}{3}$ |

5. Find the exact value of:

| | | | |
|---|---|---|---|
| a | $\sin(-210°)$ | b | $\cos(-30°)$ |
| c | $\tan(-135°)$ | d | $\cos(-420°)$ |
| e | $\cot(-60°)$ | f | $\sin(-150°)$ |
| g | $\sec(-135°)$ | h | $\text{cosec}(-120°)$ |

6. Find the exact value of:

| | | | |
|---|---|---|---|
| a | $\sin\left(-\frac{\pi}{6}\right)$ | b | $\cos\left(-\frac{3\pi}{4}\right)$ |
| c | $\tan\left(-\frac{2\pi}{3}\right)$ | d | $\sec\left(-\frac{4\pi}{3}\right)$ |
| e | $\cot\left(-\frac{3\pi}{4}\right)$ | f | $\sin\left(-\frac{7\pi}{6}\right)$ |
| g | $\cot\left(-\frac{\pi}{3}\right)$ | h | $\cos\left(-\frac{7\pi}{6}\right)$ |
| i | $\text{cosec}\left(-\frac{2\pi}{3}\right)$ | j | $\tan\left(-\frac{11\pi}{6}\right)$ |
| k | $\sec\left(-\frac{13\pi}{6}\right)$ | l | $\sin\left(-\frac{7\pi}{3}\right)$ |

7. Sketch the graphs of:

a. $y=\text{cosec}(2x)$

b. $y=\text{cotan}\left(x-\frac{\pi}{4}\right)-1$

c. $y=\sec(x-\pi)$

d. $y=\sec\left(\frac{x}{2}+\frac{\pi}{4}\right)$

e. $y=\cot(2x+\pi)$

f. $y=\text{cosec}\left(\frac{x}{2}-\pi\right)$

g. $y=\csc x-\sec(2x)$

h. $y=\cot\left(2\left(x+\frac{\pi}{4}\right)\right)$

i. $y=\csc(2x+\pi)$

j. $y=\csc\left(2x+\frac{\pi}{4}\right)+1$

k. $y=\sec(3x-\pi)-1$

## The Pythagorean Identity

We have seen a number of important relationships between trigonometric ratios. Relationships that are true for all values of θ are known as **identities**. To signal an identity (as opposed to an equation) the **equivalence** symbol is used, i.e. ≡.

For example, we can write $(x+1)^2 \equiv x^2+2x+1$ as this statement is true for all values of $x$. However, we would have to write $(x+1)^2 = x^2+1$, as this relationship is only true for some values of $x$ (which need to be determined).

One trigonometric identity is based on the unit circle.

Consider the point $P(x, y)$ on the unit circle,
$x^2+y^2 = 1$ – (1)

From the previous section, we know that

$$x = \cos\theta \quad \text{– (2)}$$

$$y = \sin\theta \quad \text{– (3)}$$

Substituting (2) and (3) into (1) we have: $(\cos\theta)^2+(\sin\theta)^2 = 1$ or

$$\sin^2\theta+\cos^2\theta=1 \quad \text{– (4)}$$

This is known as the fundamental trigonometric identity. Note that we have not used the identity symbol, i.e. we have not written $\sin^2\theta+\cos^2\theta\equiv 1$. This is because more often than not, it will be obvious from the setting as to whether a relationship is an identity or an equation. And so, there is a tendency to forgo the formal use of the identity statement and restrict ourselves to the equality statement.

By rearranging the identity we have that $\sin^2\theta=1-\cos^2\theta$ and $\cos^2\theta\equiv 1-\sin^2\theta$. Similarly we obtain the following two new identities:

Divide both sides of (4) by $\cos^2\theta$:

$$\frac{\sin^2\theta+\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta} \Leftrightarrow \frac{\sin^2\theta}{\cos^2\theta}+\frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

$$\Leftrightarrow \tan^2\theta+1 = \sec^2\theta \quad \text{– (5)}$$

Divide both sides of (4) by $\sin^2\theta$:

$$\frac{\sin^2\theta+\cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta} \Leftrightarrow \frac{\sin^2\theta}{\sin^2\theta}+\frac{\cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$$

$$\Leftrightarrow 1+\cot^2\theta = \text{cosec}^2\theta \quad \text{– (6)}$$

**Example C.8.3**

If $\cos\theta = -\frac{3}{5}$, where $\pi\le\theta\le\frac{3\pi}{2}$, find:

a $\sin\theta$ b $\tan\theta$

Problems like this can be solved by making use of a right-angled triangle, however, we now solve this question by making use of the trigonometric identities we have just developed.

a From $\sin^2\theta+\cos^2\theta = 1$ we have

$$\sin^2\theta+\left(-\frac{3}{5}\right)^2 = 1 \Leftrightarrow \sin^2\theta+\frac{9}{25} = 1$$

$$\Leftrightarrow \sin^2\theta = \frac{16}{25}$$

$$\therefore \sin\theta = \pm\frac{4}{5}$$

Now, as $\pi\le\theta\le\frac{3\pi}{2}$,

this means the angle is in the third quadrant, where the sine value is negative.

Therefore, we have that $\sin\theta = -\frac{4}{5}$.

b Using the identity $\tan\theta = \frac{\sin\theta}{\cos\theta}$,

we have $\tan\theta = \frac{(-4/5)}{(-3/5)} = \frac{4}{3}$.

**Example C.8.4**

If $\tan\theta = \frac{5}{12}$, where $\pi\le\theta\le\frac{3\pi}{2}$, find:

a $\cos\theta$ b $\text{cosec}\theta$

a From the identity $\tan^2\theta+1 = \sec^2\theta$ we have:

$$\left(\frac{5}{12}\right)^2+1 = \sec^2\theta \Leftrightarrow \sec^2\theta = \frac{25}{144}+1$$

$$\therefore \sec^2\theta = \frac{169}{144}$$

$$\therefore \sec\theta = \pm\frac{13}{12}$$

Therefore, as $\cos\theta = \frac{1}{\sec\theta} \Rightarrow \cos\theta = \pm\frac{12}{13}$.

However, $\pi\le\theta\le\frac{3\pi}{2}$, meaning that θ is in the third quadrant.

And so, the cosine is negative. That is, $\cos\theta = -\frac{12}{13}$.

Now, $\operatorname{cosec}\theta = \frac{1}{\sin\theta}$, but,

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \Leftrightarrow \sin\theta = \tan\theta\cos\theta \therefore \sin\theta = \frac{5}{12}\times -\frac{12}{13}$$
$$= -\frac{5}{13}$$

Therefore, $\operatorname{cosec}\theta = \frac{1}{(-5/13)} = -\frac{13}{5}$.

**Example C.8.5**

Simplify the following expressions.

a $\cos\theta + \tan\theta\sin\theta$ b $\frac{\cos\theta}{1+\sin\theta} - \frac{1-\sin\theta}{\cos\theta}$

a
$$\cos\theta + \tan\theta\sin\theta = \cos\theta + \frac{\sin\theta}{\cos\theta}\sin\theta$$
$$= \cos\theta + \frac{\sin^2\theta}{\cos\theta}$$
$$= \frac{\cos^2\theta + \sin^2\theta}{\cos\theta}$$
$$= \frac{1}{\cos\theta}$$
$$= \sec\theta$$

b
$$\frac{\cos\theta}{1+\sin\theta} - \frac{1-\sin\theta}{\cos\theta}$$
$$= \frac{\cos^2\theta}{(1+\sin\theta)\cos\theta} - \frac{(1-\sin\theta)(1+\sin\theta)}{(1+\sin\theta)\cos\theta}$$
$$= \frac{\cos^2\theta}{(1+\sin\theta)\cos\theta} - \frac{1-\sin^2\theta}{(1+\sin\theta)\cos\theta}$$
$$= \frac{\cos^2\theta - 1 + \sin^2\theta}{(1+\sin\theta)\cos\theta}$$
$$= \frac{(\cos^2\theta + \sin^2\theta) - 1}{(1+\sin\theta)\cos\theta}$$
$$= \frac{1-1}{(1+\sin\theta)\cos\theta}$$
$$= 0$$

**Example C.8.6**

Show that $\frac{1-2\cos^2\theta}{\sin\theta\cos\theta} = \tan\theta - \cot\theta$.

$$\text{R.H.S} = \tan\theta - \cot\theta$$
$$= \frac{\sin\theta}{\cos\theta} - \frac{\cos\theta}{\sin\theta}$$
$$= \frac{\sin^2\theta - \cos^2\theta}{\sin\theta\cos\theta}$$
$$= \frac{(1-\cos^2\theta) - \cos^2\theta}{\sin\theta\cos\theta}$$
$$= \frac{1-2\cos^2\theta}{\sin\theta\cos\theta}$$
$$= \text{L.H.S}$$

## Exercise C.8.2

1. Prove the identities.

a $\sin\theta + \cot\theta\cos\theta = \operatorname{cosec}\theta$

b $\frac{\sin\theta}{1+\cos\theta} + \frac{1+\cos\theta}{\sin\theta} = 2\operatorname{cosec}\theta$

c $\frac{\sin^2\theta}{1-\cos\theta} = 1+\cos\theta$

d $3\cos^2x - 2 = 1 - 3\sin^2x$

e $\tan^2x\cos^2x + \cot^2x\sin^2x = 1$

f $\sec\theta - \sec\theta\sin^2\theta = \cos\theta$

g $\sin^2\theta(1+\cot^2\theta) - 1 = 0$

h $\frac{1}{1-\sin\phi} + \frac{1}{1+\sin\phi} = 2\sec^2\phi$

i $\frac{\cos\theta}{1+\sin\theta} + \tan\theta = \sec\theta$

j $\frac{1-\sin\theta}{\cos\theta} = \frac{\cos\theta}{1+\sin\theta}$

k $\frac{1}{\sec x + \tan x} = \sec x - \tan x$

l $\quad \sin x + \dfrac{\cos^2 x}{1+\sin x} = 1$

m $\quad \dfrac{\sec\phi + \operatorname{cosec}\phi}{\tan\phi + \cot\phi} = \sin\phi + \cos\phi$

n $\quad \dfrac{\sin x + 1}{\cos x} = \dfrac{\sin x - \cos x + 1}{\sin x + \cos x - 1}$

o $\quad \tan x + \sec x = \dfrac{\tan x + \sec x - 1}{\tan x - \sec x + 1}$

2. Prove the following.

a $\quad (\sin x + \cos x)^2 + (\sin x - \cos x)^2 = 2$

b $\quad \sec^2\theta\operatorname{cosec}^2\theta = \sec^2\theta + \operatorname{cosec}^2\theta$

c $\quad \sin^4 x - \cos^4 x = (\sin x + \cos x)(\sin x - \cos x)$

d $\quad \sec^4 x - \sec^2 x = \tan^4 x + \tan^2 x$

e $\quad \dfrac{\sin^3 x + \cos^3 x}{\sin x + \cos x} = 1 - \sin x\cos x$

f $\quad (\cot x - \operatorname{cosec} x)^2 = \dfrac{\sec x - 1}{\sec x + 1}$

g $\quad (2b\sin x\cos x)^2 + b^2(\cos^2 x - \sin^2 x)^2 = b^2$

3. Eliminate $\theta$ from each of the following pairs.

a $\quad x = k\sin\theta, y = k\cos\theta$

b $\quad x = b\sin\theta, y = a\cos\theta$

c $\quad x = 1 + \sin\theta, y = 2 - \cos\theta$

d $\quad x = 1 - b\sin\theta, y = 2 + a\cos\theta$

e $\quad x = \sin\theta + 2\cos\theta, y = \sin\theta - 2\cos\theta$

4. a $\quad$ If $\tan\theta = \dfrac{3}{4}, \pi \le \theta \le \dfrac{3\pi}{2}$,

find: i $\cos\theta$ ii $\operatorname{cosec}\theta$

b $\quad$ If $\sin\theta = -\dfrac{3}{4}, \dfrac{3\pi}{2} \le \theta \le 2\pi$,

find: i $\sec\theta$ ii $\cot\theta$

5. Solve the following, where $0 \le \theta \le 2\pi$:

a $\quad 4\sin\theta = 3\operatorname{cosec}\theta$

b $\quad 2\cos^2\theta + \sin\theta - 1 = 0$

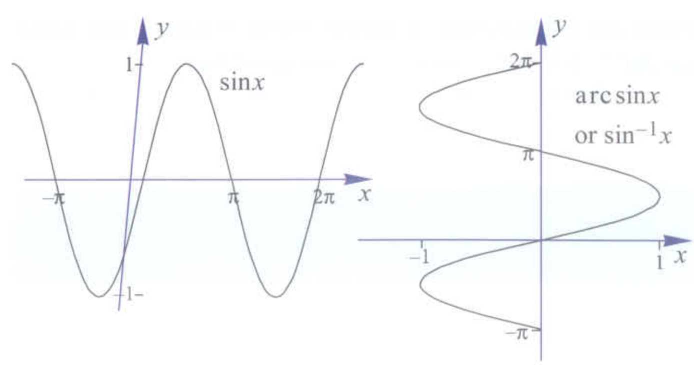
c $\quad 2 - \sin\theta = 2\cos^2\theta$

d $\quad 2\sin^2\theta = 2 + 3\cos\theta$

Extra questions

## The Inverse Sine Function

The trigonometric functions are many-to-one which means that, unless we are careful about defining domains, their inverses are not properly defined. The basic graphs of the sine function and its inverse (after reflection about the line $y = x$ for the arcsin$x$ function) are:

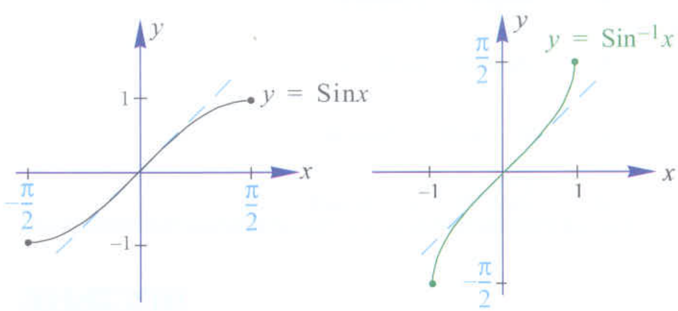



The inverse as depicted here is not a function (as it is one : many). This is inconvenient as the inverse trigonometric functions are useful. The most usual solution to this problem is to restrict the domain of the function to an interval over which it is one-to-one.

In the case of the sine function, this is usually taken as $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Though this is not the only possible choice, it is one that allows for consistency to be maintained in literature and among mathematicians. The function thus defined is written with a capital letter: $f(x) = \text{Sin}(x), x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

The graphs are:




Notice then that the domain of $\text{Sin}^{-1}x$ = range of Sin$x$ = $[-1, 1]$

and the range of $\text{Sin}^{-1}x$ = domain of Sin$x$ = $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

With these restrictions, we refer to $\text{Sin}^{-1}x$ (which is sometimes denoted by Arcsin$x$) as the principal value of arcsin$x$.

For example, $\arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}$ or $\frac{5\pi}{6}$ or $-\frac{7\pi}{6}$ or ….

However, $\text{Arcsin}\left(\frac{1}{2}\right)$ has only one value (the principal value),

so that $\text{Arcsin}\left(\frac{1}{2}\right) = \frac{\pi}{6}$.

From our fundamental identity property of inverse functions, i.e. $fof^{-1}(x) = f^{-1}of(x) = x$, we have that:

$$\text{Sin}(\text{Sin}^{-1}x) = x, -1 \leq x \leq 1 \text{ and } \text{Sin}^{-1}(\text{Sin}x) = x, -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$$

Therefore, $\text{Sin}(\text{Sin}^{-1}x) = x = \text{Sin}^{-1}(\text{Sin}x)$ only if $-1 \leq x \leq 1$.

This then means that sometimes we can provide a meaningful interpretation to expressions such as $\sin(\text{Sin}^{-1}x)$ & $\text{Sin}^{-1}(\sin x)$ – as long as we adhere to the relevant restrictions.

**Example C.8.7**

Give the exact value of:

| | | | |
|---|---|---|---|
| a | $\text{Sin}^{-1}\frac{1}{2}$ | b | $\text{Arcsin}\left(-\frac{\sqrt{3}}{2}\right)$ |
| c | $\text{Sin}^{-1}(1.3)$ | d | $\text{Sin}^{-1}(\sin\pi)$ |

a As $\frac{1}{2} \in [-1, 1] \Rightarrow \text{Sin}^{-1}\frac{1}{2}$ exists.

Therefore, $\text{Sin}^{-1}\frac{1}{2} = \frac{\pi}{6}$.

b As $-\frac{\sqrt{3}}{2} \in [-1, 1] \Rightarrow \text{Arcsin}\left(-\frac{\sqrt{3}}{2}\right)$ exists.

Now, $\text{Arcsin}\left(-\frac{\sqrt{3}}{2}\right) = -\text{Arcsin}\left(\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}$.

c As $1.3 \notin [-1, 1] \Rightarrow \text{Sin}^{-1}(1.3)$ does **not** exist.

d As $\sin\pi \in [-1, 1] \Rightarrow \text{Sin}^{-1}(\sin\pi)$ exists.

So, $\text{Sin}^{-1}(\sin\pi) = \text{Sin}^{-1}(0) = 0$.

Note that $\text{Sin}^{-1}(\sin\pi) \neq \pi$! Why?

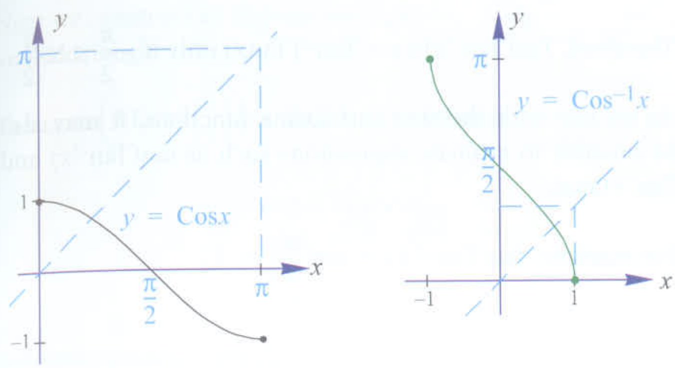
## The Inverse Cosine Function

For similar reasons as those for the sine function, the cosine function, $\cos x$, $x \in ]-\infty,\infty[$ being a many-to-one function, with its inverse, arccos$x$, $-1 \leq x \leq 1$ (or $\cos^{-1}x$, $-1 \leq x \leq 1$) needs to be restricted to the domain $[0,\pi]$, to produce a function that is one-to-one.

The function $y = \text{Cos}x, x \in [0,\pi], -1 \leq y \leq 1$ (with a capital 'C') will have the inverse function defined as:

$$f(x) = \text{Cos}^{-1}x, -1 \leq x \leq 1, -1 \leq y \leq \pi$$

The graphs of these functions are:




Notice that the domain of $\text{Cos}^{-1}x$ = range of $\text{Cos}x = [-1, 1]$ and the range of $\text{Cos}^{-1}x$ = domain of $\text{Cos}x = [0,\pi]$.

When these restrictions are adhered to, we refer to $\text{Cos}^{-1}x$ (which is sometimes denoted by Arccos$x$) as the principal value of arccos$x$.

From our fundamental identity property of inverse functions,

i.e. $f \circ f^{-1}(x) = f^{-1} \circ f(x) = x$ , we have that:

$\text{Cos}(\text{Cos}^{-1}x) = x,\ -1 \le x \le 1$ and $\text{Cos}^{-1}(\text{Cos}x) = x,\ -0 \le x \le \pi$

Therefore, $\text{Cos}(\text{Cos}^{-1}x) = x = \text{Cos}^{-1}(\text{Cos}x)$ only if $0 \le x \le 1$.

This means that we can provide a meaningful interpretation of expressions such as $\cos(\text{Cos}^{-1}x)$ and $\text{Cos}^{-1}(\text{Cos}x)$– as long as we adhere to the relevant restrictions.

Note also that in this case, $\text{Cos}^{-1}(-x) \neq -\text{Cos}^{-1}(x)$ .

**Example C.8.8**

Given the exact value of:

a $\quad \text{Cos}^{-1}\frac{1}{2}$ $\qquad$ b $\quad \text{Arccos}\left(-\frac{\sqrt{3}}{2}\right)$

c $\quad \text{Cos}^{-1}\left(\cos\left(\frac{3\pi}{2}\right)\right)$

a $\quad$ As $\frac{1}{2} \in [-1, 1] \Rightarrow \text{Cos}^{-1}\frac{1}{2}$ exists.

Therefore, $\text{Cos}^{-1}\frac{1}{2} = \frac{\pi}{3}$.

b $\quad$ As $-\frac{\sqrt{3}}{2} \in [-1, 1] \Rightarrow \text{Cos}^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ exists.

Let $y = \text{Cos}^{-1}\left(-\frac{\sqrt{3}}{2}\right)$, then, $\text{Cos}y = -\frac{\sqrt{3}}{2},\ 0 \le y \le \pi$.

$$\Leftrightarrow y = \pi - \frac{\pi}{6}$$

$$= \frac{5\pi}{6}$$

c $\quad$ As $\cos\left(\frac{3\pi}{2}\right) \in [-1, 1] \Rightarrow \text{Cos}^{-1}\left(\cos\left(\frac{3\pi}{2}\right)\right)$ exists.

$\text{Cos}^{-1}\left(\cos\left(\frac{3\pi}{2}\right)\right) = \text{Cos}^{-1}(0) = \frac{\pi}{2}$.

Notice that $\text{Cos}^{-1}\left(\cos\left(\frac{3\pi}{2}\right)\right) \neq \frac{3\pi}{2}$.

**Example C.8.9**

Give the exact value of:

a $\quad \sin\left(\text{Arccos}\left(\frac{1}{\sqrt{2}}\right)\right)$ $\qquad$ b $\quad \cos\left(\text{Sin}^{-1}\left(\frac{1}{4}\right)\right)$

c $\quad \sin\left(\frac{\pi}{2} - \text{Cos}^{-1}\left(\frac{3}{4}\right)\right)$

a

Let $\text{Arccos}\left(\frac{1}{\sqrt{2}}\right) = x \therefore$ as $\frac{1}{\sqrt{2}} \in [0, 1] \Rightarrow \text{Arccos}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$

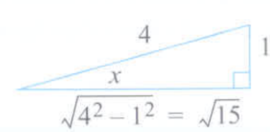
Then, $\sin\left(\text{Arccos}\left(\frac{1}{\sqrt{2}}\right)\right) = \sin(x) = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$.

Let $\text{Sin}^{-1}\left(\frac{1}{4}\right) = x \therefore$ as $\frac{1}{4} \in [-1, 1] \Rightarrow \text{Sin}^{-1}\left(\frac{1}{4}\right)$ exists.

b $\quad$ However, this time we cannot obtain an exact value for $x$, so we make use of a right-angled triangle:

Therefore, from the triangle

we have that $\cos x = \frac{\sqrt{15}}{4}$.




i.e. $\cos\left(\text{Sin}^{-1}\left(\frac{1}{4}\right)\right) = \cos x = \frac{\sqrt{15}}{4}$.

c Let $\text{Cos}^{-1}\left(\frac{3}{4}\right) = \theta \therefore$ as $\frac{3}{4} \in [-1, 1] \Rightarrow \text{Cos}^{-1}\left(\frac{3}{4}\right)$

Then, $\sin\left(\frac{\pi}{2} - \text{Cos}^{-1}\left(\frac{3}{4}\right)\right) = \sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$.

Therefore, $\sin\left(\frac{\pi}{2} - \text{Cos}^{-1}\left(\frac{3}{4}\right)\right) = \cos\left(\text{Cos}^{-1}\left(\frac{3}{4}\right)\right) = \frac{3}{4}$

# The Inverse Tangent Function

The tangent function can be made one : one by restricting its domain to the open interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

$f(x) = \text{Tan}(x), x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

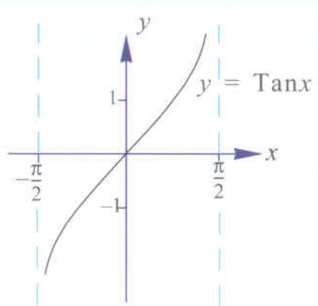
The function:

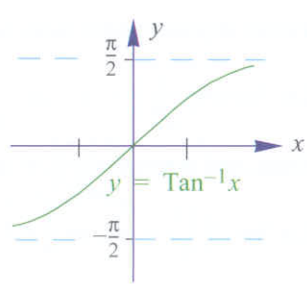
$y = \text{Tan}(x), x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), -\infty < y < \infty$, (with a capital 'T')

will have the inverse function defined as:

$f(x) = \text{Tan}^{-1}(x), , -\infty < x < \infty$.

The graphs of these functions are:




Notice then that the domain of $\text{Tan}^{-1}x$ = range of Tan$x$ = $(-\infty, \infty)$ and the range of $\text{Tan}^{-1}x$ = domain of Tan$x$ = $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

When these restrictions are adhered to, we refer to $\text{Tan}^{-1}x$ (which is sometimes denoted by Arctan$x$) as the principal value of arctan$x$.

$\text{Tan}(\text{Tan}^{-1}x) = x, -\infty \le x \le \infty$ and $\text{Tan}^{-1}(\text{Tan}x) = x, -\frac{\pi}{2} \le x \le \frac{\pi}{2}$

From our fundamental identity property of inverse functions, i.e. $f o f^{-1}(x) = f^{-1} o f(x) = x$, we have that

Therefore, $\text{Tan}(\text{Tan}^{-1}x) = x = \text{Tan}^{-1}(\text{Tan}x)$ only if $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$.

As we saw with the sine and cosine functions, it **may** also be possible to evaluate expressions such as $\tan(\text{Tan}^{-1}x)$ and $\text{Tan}^{-1}(\tan x)$.

For example, $\tan(\text{Tan}^{-1}1) = \tan\left(\frac{\pi}{4}\right) = 1$,

however, $\text{Tan}^{-1}\left(\tan\frac{2\pi}{3}\right) = \text{Tan}^{-1}(-\sqrt{3}) = -\frac{\pi}{3}$.

Note also that $\text{Tan}^{-1}(-x) = -\text{Tan}(x)$

### Example C.8.10

Give the exact value of:

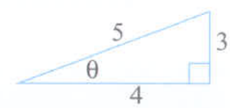
a $\tan\left(\text{Sin}^{-1}\left(-\frac{3}{5}\right)\right)$ b $\sin\left(2\text{Tan}^{-1}\left(\frac{1}{3}\right)\right)$

a As $-\frac{3}{5} \in [-1, 1] \Rightarrow \text{Sin}^{-1}\left(-\frac{3}{5}\right)$ exists.

Then, we let $\theta = \text{Sin}^{-1}\left(\frac{3}{5}\right)$, so that $\text{Sin}\theta = \frac{3}{5}$.

Next we construct an appropriate right-angled triangle:

So, $\tan\left(\text{Sin}^{-1}\left(-\frac{3}{5}\right)\right) = \tan\left(-\text{Sin}^{-1}\left(\frac{3}{5}\right)\right)$

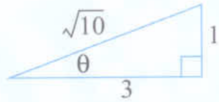



$= \tan(-\theta) = -\tan\theta = -\frac{3}{4}$

b As $\frac{1}{3} \in (-\infty, \infty) \Rightarrow \text{Tan}^{-1}\left(\frac{1}{3}\right)$ exists.

Let $\text{Tan}^{-1}\left(\frac{1}{3}\right) = \theta \therefore \text{Tan}\theta = \frac{1}{3}$.

Next we construct an appropriate right-angled triangle:




Then, $\sin\left(2\operatorname{Tan}^{-1}\left(\frac{1}{3}\right)\right) = \sin 2\theta = 2\sin\theta\cos\theta$

$$= 2 \times \frac{1}{\sqrt{10}} \times \frac{3}{\sqrt{10}}$$

$$= \frac{3}{5}$$

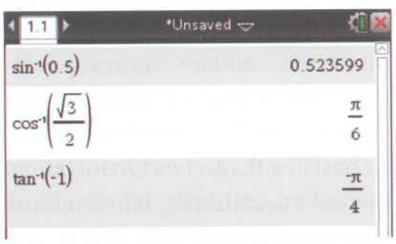
It is these restricted functions that are programmed into most calculators, spreadsheets etc.

If the calculator is set in radian mode, some sample calculations are:

$\operatorname{Sin}^{-1}\frac{1}{2} = \frac{\pi}{6}$

$\operatorname{Cos}^{-1}\frac{\sqrt{3}}{2} = \frac{\pi}{6}$

$\operatorname{Tan}^{-1}-1 = -\frac{\pi}{4}$

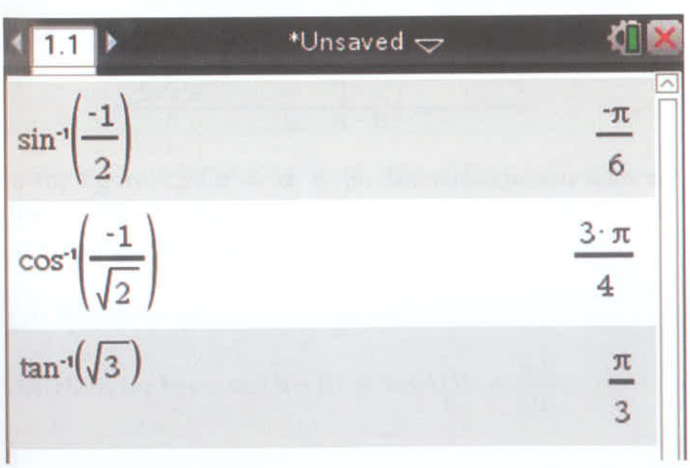



**Example C.8.11**

Find the principal values of the following.

a $\operatorname{Sin}^{-1}\left(-\frac{1}{2}\right)$ b $\operatorname{Cos}^{-1}\left(-\frac{1}{\sqrt{2}}\right)$

c $\operatorname{Tan}^{-1}\sqrt{3}$




## Exercise C.8.3

1. Find the principal values of the following, giving answers in radians.

a $\operatorname{Tan}^{-1}1$ b $\operatorname{Arc}\sin 1$

c $\operatorname{Arc}\cos -1$

d $\operatorname{Sin}^{-1}\frac{\sqrt{3}}{2}$ e $\operatorname{Cos}^{-1}\frac{1}{\sqrt{2}}$

f $\operatorname{Tan}^{-1}-\sqrt{3}$ g $\operatorname{Tan}^{-1}2$

h $\operatorname{Sin}^{-1}-0.7$ i $\operatorname{Arc}\tan 0.1$

j $\operatorname{Arc}\cos 0.3$ k $\operatorname{Sin}^{-1}-0.6$

l $\operatorname{Tan}^{-1}5$ m $\operatorname{Cos}^{-1}3$

n $\operatorname{Tan}^{-1}-30$ o $\operatorname{Sin}^{-1}\left(\frac{7}{8}\right)$

2. Solve the following equations, giving exact answers.

a $\operatorname{Arctan}x = \frac{3\pi}{4}$

b $\operatorname{Arcsin}(2x) = \frac{\pi}{3}$

c $\operatorname{Arccos}(3x) = \frac{5\pi}{4}$

3. Prove:

a $\operatorname{Arctan}(4) - \operatorname{Arctan}\left(\frac{3}{5}\right) = \frac{\pi}{4}$

b $\operatorname{Sin}^{-1}\left(\frac{4}{5}\right) + \operatorname{Sin}^{-1}\left(-\frac{4}{5}\right) = 0$

4. Solve for $x$, where:

$\operatorname{Arctan}(3x) - \operatorname{Arctan}(2x) = \operatorname{Arctan}\left(\frac{1}{5}\right)$

5. Find the exact value of:

a $\sin\left[\frac{\pi}{2} - \operatorname{Cos}^{-1}\left(\frac{2}{3}\right)\right]$

b $\cos\left[\frac{\pi}{2} + \operatorname{Sin}^{-1}\left(-\frac{1}{3}\right)\right]$

c $\cos[\text{Tan}^{-1}(-\sqrt{3})]$

d $\tan\left(\text{Cos}^{-1}\left(\frac{4}{5}\right)\right)$

e $\sec\left(\text{Sin}^{-1}\left(-\frac{1}{3}\right)\right)$

f $\cot(\text{Tan}^{-1}(-1))$

Extra example and questions

Answers

# C.9 Further Identities

AHL 3.10

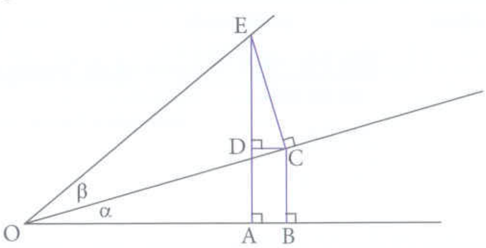
## Compound Angle Identities

As we have seen in the previous section, there are numerous trigonometric identities. However, they were all derived from the fundamental identities. In this section we develop some more fundamental identities (which will lead us to more identities). These fundamental identities are known as compound angle identities. That is, they are identities that involve the sine, cosine and tangent of the sum and difference of two angles.

We start with the sine of the sum of two angles, $\sin(\alpha+\beta)$:

The Diploma Course does not expect students to prove this result. It is included for the sake of completeness.

A commonly given proof of these identities is only valid for acute angles:




In the figure, $\angle AOE = \alpha + \beta$. The construction lines are drawn with the right angles indicated. Since $\angle DCO = \alpha$ (alternate angles) and $\angle DCE = 90° - \alpha$, it follows that

$\angle AOE = \alpha + \beta$.

Therefore, we have, $\sin(\alpha+\beta) = \sin AOE = \dfrac{AE}{OE}$

$$= \frac{AD+DE}{OE}$$
$$= \frac{AD}{OE}+\frac{DE}{OE}$$
$$= \frac{BC}{OE}+\frac{DE}{OE}$$
$$= \frac{BC}{OC}\times\frac{OC}{OE}+\frac{DE}{EC}\times\frac{EC}{OE}$$
$$= \sin\alpha\times\cos\beta+\cos\alpha\times\sin\beta$$

It is now possible to prove the difference formula, replacing $\beta$ by $-\beta$ we have:

$$\begin{aligned}\sin(\alpha-\beta) &= \sin(\alpha+(-\beta))\\ &= \sin\alpha\cos(-\beta)+\cos\alpha\sin(-\beta)\\ &= \sin\alpha\cos\beta-\cos\alpha\sin\beta\end{aligned}$$

($\cos(-\beta) = \cos\beta$ and $\sin(-\beta) = -\sin\beta$)

And so we have the addition and difference identities for sine:

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$$
$$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$$

A similar identity can be derived for the cosine function (using the same diagram):

$$\cos(\alpha+\beta) = \frac{OA}{OE} = \frac{OB-AB}{OE}$$
$$= \frac{OB}{OE}-\frac{AB}{OE}$$
$$= \frac{OB}{OE}-\frac{CD}{OE}$$
$$= \frac{OB}{OC}\times\frac{OC}{OE}-\frac{CD}{EC}\times\frac{EC}{OE}$$
$$= \cos\alpha\cos\beta-\sin\alpha\sin\beta$$

Similarly, from this and replacing $\beta$ by $-\beta$ we have that $\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$.

And so we have the addition and difference identities for cosine:

$$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$$
$$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$$

Also, the tangent addition identity can be proved as follows:

Using $\frac{\sin\theta}{\cos\theta} = \tan\theta$

$$\begin{aligned}\tan(\alpha+\beta) &= \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}\\ &= \frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta-\sin\alpha\sin\beta}\\ &= \frac{\dfrac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\dfrac{\cos\alpha\cos\beta-\sin\alpha\sin\beta}{\cos\alpha\cos\beta}}\\ &= \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}\end{aligned}$$

Again, if we replace $\beta$ by $-\beta$ we have:

$$\tan(\alpha-\beta) = \frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}.$$

And so we have the addition and difference identities for tangent:

$$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$$
$$\tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$$

As a special case of the compound identities we have obtained so far, we have a set of identities known as the **double-angle identities**.

Using the substitution $\theta = \alpha = \beta$ we obtain the identities:

$$\sin 2\theta=2\sin\theta\cos\theta$$
$$\cos 2\theta=\cos^2\theta-\sin^2\theta$$

i.e. substituting $\theta = \alpha = \beta$ into:

$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$ we obtain

$$\begin{aligned}\sin(\theta+\theta) &= \sin\theta\cos\theta+\cos\theta\sin\theta\\ \therefore \sin 2\theta &= 2\sin\theta\cos\theta\end{aligned}$$

Similarly, substituting $\theta = \alpha = \beta$ into:

$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$ we obtain

$$\begin{aligned}\cos(\theta+\theta) &= \cos\theta\cos\theta-\sin\theta\sin\theta\\ \therefore \cos 2\theta &= \cos^2\theta-\sin^2\theta\end{aligned}$$

The second of these can be further developed to give:

$$\cos 2\theta = \cos^2\theta-\sin^2\theta = \cos^2\theta-(1-\cos^2\theta) = 2\cos^2\theta-1$$

and

$$\cos 2\theta = \cos^2\theta-\sin^2\theta = (1-\sin^2\theta)-\sin^2\theta = 1-2\sin^2\theta$$

Finally, we have a double-angle identity for the tangent:

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$$

## Summary of double-angle identities

$$\begin{aligned}\sin 2\theta&=2\sin\theta\cos\theta\\ \cos 2\theta&=\cos^2\theta-\sin^2\theta\\ &=2\cos^2\theta-1\\ &=1-2\sin^2\theta\\ \tan 2\theta&=\frac{2\tan\theta}{1-\tan^2\theta}\end{aligned}$$

We have seen how trigonometric identities can be used to solve equations, simplify expressions and to prove further identities. We now illustrate this using the new set of identities.

**Example C.9.1**

Simplify the expression: $\frac{\sin 3\alpha}{\sin\alpha} - \frac{\cos 3\alpha}{\cos\alpha}$.

$$\begin{aligned}\frac{\sin 3\alpha}{\sin\alpha}-\frac{\cos 3\alpha}{\cos\alpha} &= \frac{\sin 3\alpha\cos\alpha-\cos 3\alpha\sin\alpha}{\sin\alpha\cos\alpha}\\ &= \frac{\sin(3\alpha-\alpha)}{\sin\alpha\cos\alpha}\\ &= \frac{\sin 2\alpha}{\frac{1}{2}\sin 2\alpha}\\ &= 2\end{aligned}$$

**Example C.9.2**

Prove the identity $\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha$.

$$\cos 3\alpha = \cos(2\alpha + \alpha)$$
$$= \cos 2\alpha \cos\alpha - \sin 2\alpha \sin\alpha$$
$$= (2\cos^2\alpha - 1)\cos\alpha - 2\sin\alpha\cos\alpha\sin\alpha$$
$$= 2\cos^3\alpha - \cos\alpha - 2\sin^2\alpha\cos\alpha$$
$$= 2\cos^3\alpha - \cos\alpha - 2(1 - \cos^2\alpha)\cos\alpha$$
$$= 2\cos^3\alpha - \cos\alpha - 2\cos\alpha + 2\cos^3\alpha$$
$$= 4\cos^3\alpha - 3\cos\alpha$$

### Example C.9.3

Using a compound identity, show that $\cos\left(\frac{3\pi}{2} - \theta\right) = -\sin\theta$

$$\text{L.H.S} = \cos\left(\frac{3\pi}{2} - \theta\right)$$
$$= \cos\left(\frac{3\pi}{2}\right)\cos\theta + \sin\left(\frac{3\pi}{2}\right)\sin\theta$$
$$= 0 \times \cos\theta + (-1) \times \sin\theta$$
$$= -\sin\theta$$
$$= \text{R.H.S}$$

### Example C.9.4

Find the exact value of:

a $\cos 15°$  b $\tan\frac{5\pi}{12}$

a

$$\cos 15° = \cos(45° - 30°) = \cos 45° \cos 30° + \sin 45° \sin 30°$$
$$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \times \frac{1}{2}$$
$$= \frac{\sqrt{3}+1}{2\sqrt{2}}$$

b

$$\tan\frac{5\pi}{12} = \tan\left(\frac{3\pi}{12} + \frac{2\pi}{12}\right) = \frac{\tan\frac{3\pi}{12} + \tan\frac{2\pi}{12}}{1 - \tan\frac{3\pi}{12}\tan\frac{2\pi}{12}} = \frac{\tan\frac{\pi}{4} + \tan\frac{\pi}{6}}{1 - \tan\frac{\pi}{4}\tan\frac{\pi}{6}}$$
$$= \frac{1 + \frac{1}{\sqrt{3}}}{1 - 1 \times \frac{1}{\sqrt{3}}} = \frac{\sqrt{3}+1}{\sqrt{3}-1}$$

### Example C.9.5

Prove that $\frac{\sin 2\phi + \sin\phi}{\cos 2\phi + \cos\phi + 1} = \tan\phi$ .

$$\text{L.H.S} = \frac{\sin 2\phi + \sin\phi}{\cos 2\phi + \cos\phi + 1} = \frac{2\sin\phi\cos\phi + \sin\phi}{2\cos^2\phi - 1 + \cos\phi + 1}$$
$$= \frac{\sin\phi(2\cos\phi + 1)}{\cos\phi(2\cos\phi + 1)}$$
$$= \frac{\sin\phi}{\cos\phi}$$
$$= \tan\phi$$
$$= \text{R.H.S}$$

## Exercise C.9.1

1. Expand the following.

a $\sin(\alpha + \phi)$  b $\cos(3\alpha + 2\beta)$

c $\sin(2x - y)$  d $\cos(\phi - 2\alpha)$

e $\tan(2\theta - \alpha)$  f $\tan(\phi - 3\omega)$

2. Simplify the following.

a $\sin 2\alpha \cos 3\beta - \sin 3\beta \cos 2\alpha$

b $\cos 2\alpha \cos 5\beta - \sin 2\alpha \sin 5\beta$

c $\sin x \cos 2y + \sin 2y \cos x$

d $\cos x \cos 3y + \sin x \sin 3y$

e $\frac{\tan 2\alpha - \tan\beta}{1 + \tan 2\alpha \tan\beta}$

f $\dfrac{\tan(x-y)+\tan y}{1-\tan(x-y)\tan y}$

g $\dfrac{1-\tan\phi}{1+\tan\phi}$

h $\dfrac{1}{\sqrt{2}}\sin(\alpha+\beta)+\dfrac{1}{\sqrt{2}}\cos(\alpha+\beta)$

3. Given that $\sin\theta = \frac{4}{5}, 0 \le \theta \le \frac{\pi}{2}$ and $\cos\phi = -\frac{5}{13}, \pi \le \phi \le \frac{3\pi}{2}$, evaluate:

a $\sin(\theta+\phi)$

b $\cos(\theta+\phi)$

c $\tan(\theta-\phi)$

4. Given that $\sin\theta = -\frac{3}{5}, \pi \le \theta \le \frac{3\pi}{2}$ and $\cos\phi = -\frac{12}{13}, \pi \le \phi \le \frac{3\pi}{2}$, evaluate:

a $\sin(\theta-\phi)$

b $\cos(\theta-\phi)$

c $\tan(\theta+\phi)$

5. Given that $\sin\theta = -\frac{5}{6}, \frac{3\pi}{2} \le \theta \le 2\pi$, evaluate:

a $\sin 2\theta$

b $\cos 2\theta$

c $\tan 2\theta$

d $\sin 4\theta$

6. Given that $\tan x = -3, \frac{\pi}{2} \le x \le \pi$, evaluate:

a $\sin 2x$

b $\cos 2x$

c $\tan 2x$

d $\tan 4x$

7. Find the exact value of:

a $\sin\frac{5\pi}{12}$ b $\sin 105°$

c $\cos\frac{11\pi}{12}$ d $\tan 165°$

8. Given that $\tan x = \frac{a}{b}, \pi \le x \le \frac{3\pi}{2}$, evaluate

a $\sin 2x$ b $\operatorname{cosec} 2x$

c $\cos 4x$ d $\tan 2x$

9. Prove the following identities.

a $\cot x - \cot 2x = \operatorname{cosec} 2x$

b $\sin(x+y)\sin(x-y) = \sin^2 x - \sin^2 y$

c $\sec^2 x = 1 + \tan^2 x$

d $\tan(\theta+\phi)+\tan(\theta-\phi) = \dfrac{2\sin 2\theta}{\cos 2\theta + \cos 2\phi}$

e $\cos^4\alpha - \sin^4\alpha = 1 - 2\sin^2\alpha$

f $\dfrac{1}{\sin y\cos y} - \dfrac{\cos y}{\sin y} = \tan y$

g $\dfrac{1+\cos 2y}{\sin 2y} = \dfrac{\sin 2y}{1-\cos 2y}$

h $\csc\left(\theta+\dfrac{\pi}{2}\right) = \sec\theta$

i $\cos 3x = \cos x - 4\sin^2 x\cos x$

j $\dfrac{1+\sin 2\theta}{\cos 2\theta} = \dfrac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}$

k $(\cot x + \csc x)^2 = \dfrac{1+\cos x}{1-\cos x}$

l $\sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha$

m $\cos 2x = \dfrac{1-\tan^2 x}{1+\tan^2 x}$

n $2\cot\theta\sin^2\theta = \sin 2\theta$

o $\tan\left(\dfrac{\phi}{2}\right) = \csc\phi - \cot\phi$

Extra examples and questions

Answers

# C.10 Trigonometric Functions

AHL 3.11

## Symmetry

This chapter will look at the symmetric properties of the trigonometric functions and the implications that they have for the relationships between them and their use in modelling.

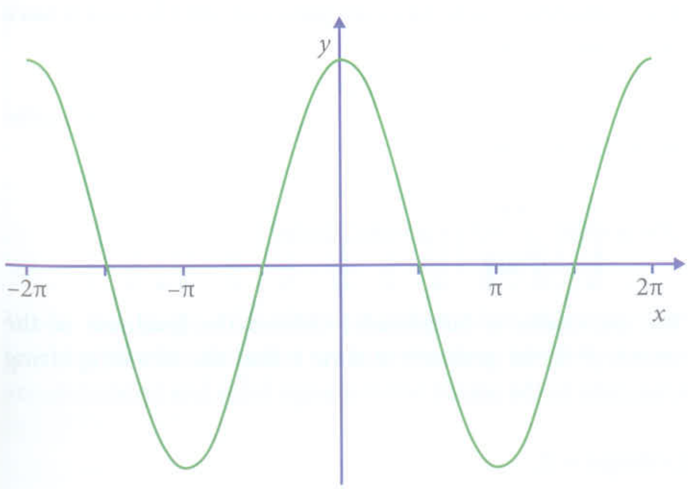
**Example C.10.1**

Examine these identities graphically:

a $\quad \cos x \equiv -\sin\left(x + \dfrac{3\pi}{2}\right)$

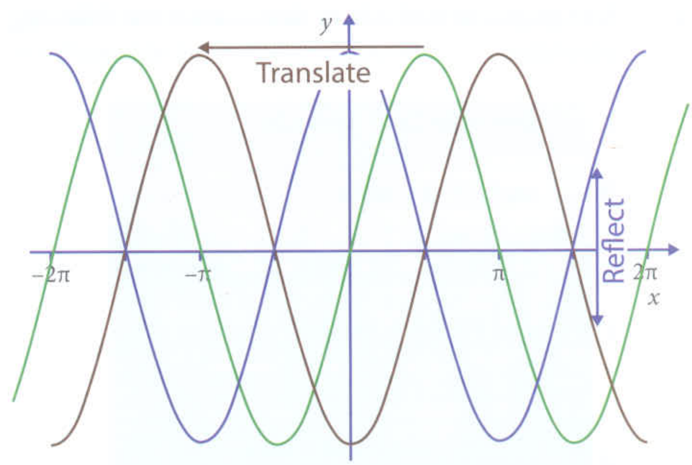
b $\quad (\sin x + \cos x)^2 + (\sin x - \cos x)^2 = 2$

a The cosine graph is:




The right hand side of the identity is the sine graph with the transformations:

1. $\dfrac{3\pi}{2}$ to the left.
2. Reflection in the $x$-axis.




The original sine graph (green) is translated $\dfrac{3\pi}{2}$ to the left to give the brown curve.

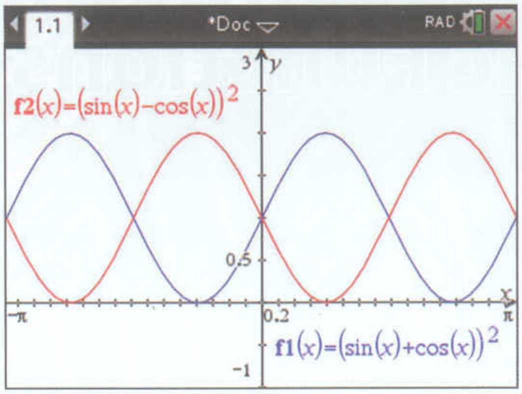
This is then reflected in the $x$-axis (inverted) to give the blue curve.

This is identical to the cosine curve (in the left hand column).

You might like to discuss whether this constitutes a proof.

b We will present this answer as it might appear if you use a graphic calculator.

$$(\sin x + \cos x)^2 + (\sin x - \cos x)^2 = 2$$




Addition of ordinates $y$-coordinates - (either by inspection or using the calculator) leads to the required result.

### Exercise C.10.1

1. Use graphical methods to demonstrate the following identities.

a $\cot x - \cot 2x = \operatorname{cosec} 2x$

b $\sec^2 x = 1 + \tan^2 x$

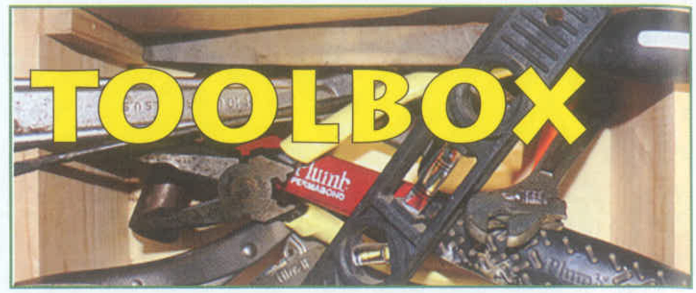
c $\csc\left(\theta + \frac{\pi}{2}\right) = \sec\theta$

d $(\cot x + \csc x)^2 = \frac{1 + \cos x}{1 - \cos x}$

e $\cos 2x = \frac{1 - \tan^2 x}{1 + \tan^2 x}$

f $2\cot\theta\sin^2\theta = \sin 2\theta$

g $\tan\left(\frac{\phi}{2}\right) = \csc\phi - \cot\phi$




## Modelling using Trigonometric Functions

Many natural phenomena are periodic. The trigonometric functions are ideal for modelling such functions. They are, therefore good sources of topics for extended investigations. We will sketch out two suggestions.

### Music

We hear musical notes because musical instruments (including animal voice-boxes) vibrate and transmit the vibrations to the air. These then travel to our ear where they make the eardrum vibrate in the same way. The inner ear then decodes the vibration and sends messages to the brain that then 'hears' the sound.

The process is, however, complex as a musical note is almost never a pure sine wave (which sounds very dull!).

The subtleties of musical notes are mainly due to what are known as 'harmonics'. A guitar string that is plucked vibrates mainly like this:

This is known as the 'fundamental' note. For audible notes, these have frequencies from 20 to 20 000 Hertz (cycles per second).

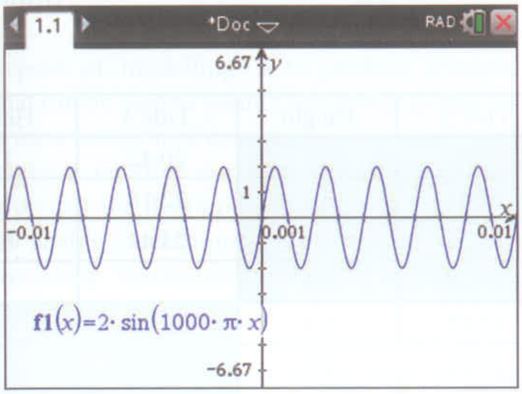
If a fundamental note has a frequency of 500 Hz then it has a period of $^1/_{500}$ sec.

This means, using $\tau = \frac{2\pi}{n} \Rightarrow n = \frac{2\pi}{\tau}$, that a possible modelling function is:

$$P = a \times \sin\left(\frac{2\pi}{1/500}t\right) = a \times \sin(1000\pi t)$$

The parameter $a$ (amplitude) defines the loudness of the sound. $P$ is the pressure at time $t$ that the vibrating string transmits to the air.

Letting $a = 2$.



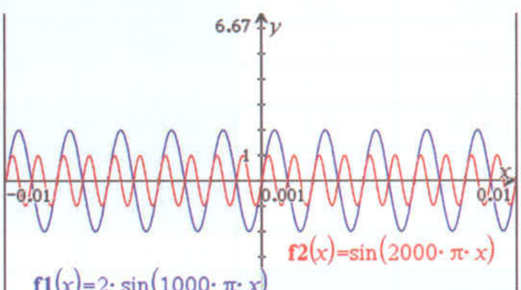

The horizontal scale runs from –0.01 to 0.01 sec. There are 10 complete periods in 0.02 sec so there are 10 ÷ 0.02 = 500 cycles per second (as required).

But this is not all. The string usually vibrates at other frequencies at the same time. This happens naturally but also because the string is not plucked in the middle. The first 'harmonic' is this vibration:

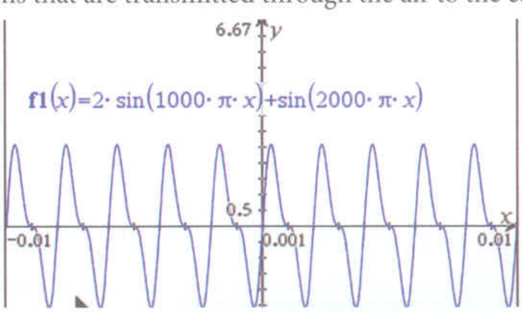
This is at twice the frequency and is usually not as loud as the fundamental. It is important to note that they sound together.

If we add the function: $P = 1\times\sin\left(\frac{2\pi}{\frac{1}{1000}}t\right) = 1\times\sin(2000\pi t)$

with twice the frequency and half the amplitude, we get:




The two notes sound together and it is the sum of the two functions that are transmitted through the air to the ear.




When this single vibration is transmitted to the inner ear (cochlea) the two hairs that are tuned to these two frequencies are stimulated and send signals to the brain.

When we perform this breakdown of a complex periodic function into its component sine functions, we call it 'Fourier Analysis' and it is difficult!

The brain then performs an analysis of the signals.

Next time you listen to some music, think about the complexities of what is going on in your brain. A single vibration hits your ear and yet you are able to hear the person singing and numerous separate instruments - a prodigious feat of computation.

If you would like to undertake an investigation of the mathematics of real musical sounds, you will need to capture graphs of their waveforms.

There are some 'apps' that will do this for you. Search the app store using the search word 'oscilloscope'.

An oscilloscope is a device that displays variable signals as graphs.

## Tides

The data in the remainder of this chapter are tide tables for San Diego Harbour (California, USA).

They were derived from the website:

https://tides4fishing.com/us/california/san-diego

## Raw Data

| Date | Tide 1 | Height | Tide 2 | Height | Tide 3 | Height | Tide 4 | Height |
|---|---|---|---|---|---|---|---|---|
| 1 | 1:15 | 4.0 | 6:56 | 1.9 | 13:38 | 5.2 | 20:42 | 1.6 |
| 2 | 2:47 | 3.5 | 7:45 | 2.4 | 14:44 | 5.2 | 22:18 | 1.3 |
| 3 | 4:51 | 3.4 | 9:15 | 2.8 | 16:04 | 5.4 | 23:42 | 0.8 |
| 4 | 6:27 | 3.7 | 10:56 | 2.8 | 17:20 | 5.8 | | |
| 5 | 0:44 | 0.1 | 7:22 | 4.2 | 12:12 | 2.5 | 18:24 | 6.3 |
| 6 | 1:33 | -0.4 | 8:03 | 4.6 | 13:11 | 2.0 | 19:20 | 6.7 |
| 7 | 2:17 | -0.8 | 8:04 | 5.1 | 14:02 | 1.5 | 20:10 | 7.0 |
| 8 | 2:57 | -1.0 | 9:16 | 5.5 | 14:50 | 1.0 | 20:58 | 7.1 |
| 9 | 3:36 | -0.9 | 9:51 | 5.8 | 15:36 | 0.7 | 21:43 | 7.0 |
| 10 | 4:13 | -0.6 | 10:27 | 6.0 | 16:22 | 0.5 | 22:29 | 6.5 |
| 11 | 4:49 | -0.2 | 11:03 | 6.0 | 17:09 | 0.5 | 23:15 | 5.9 |
| 12 | 5:25 | 0.5 | 11:41 | 5.9 | 17:59 | 0.6 | | |
| 13 | 0:04 | 5.2 | 6:01 | 1.2 | 12:20 | 5.7 | 18:54 | 0.9 |
| 14 | 1:00 | 4.4 | 6:39 | 1.9 | 13:04 | 5.4 | 20:00 | 1.2 |
| 15 | 2:14 | 3.8 | 7:23 | 2.5 | 13:57 | 5.1 | 21:23 | 1.3 |
| 16 | 4:10 | 3.5 | 8:32 | 3.0 | 15:09 | 4.9 | 22:58 | 1.2 |
| 17 | 6:24 | 3.7 | 10:20 | 3.2 | 16:34 | 4.8 | | |
| 18 | 0:11 | 1.0 | 7:18 | 4.0 | 11:52 | 3.1 | 17:46 | 5.0 |
| 19 | 1:00 | 0.7 | 7:47 | 4.3 | 12:47 | 2.7 | 18:39 | 5.3 |
| 20 | 1:36 | 0.5 | 8:09 | 4.6 | 13:25 | 2.3 | 19:21 | 5.5 |
| 21 | 2:06 | 0.3 | 8:29 | 4.8 | 13:57 | 1.9 | 19:57 | 5.8 |
| 22 | 2:34 | 0.2 | 8:50 | 5.0 | 14:28 | 1.6 | 20:30 | 5.9 |
| 23 | 3:00 | 0.2 | 9:12 | 5.3 | 14:59 | 1.2 | 21:03 | 5.9 |
| 24 | 3:25 | 0.2 | 9:36 | 5.5 | 15:31 | 1.0 | 21:36 | 5.8 |
| 25 | 3:51 | 0.4 | 10:01 | 5.7 | 16:05 | 0.8 | 22:10 | 5.6 |
| 26 | 4:17 | 0.7 | 10:28 | 5.8 | 16:41 | 0.7 | 22:47 | 5.3 |
| 27 | 4:44 | 1.0 | 10:56 | 5.9 | 17:21 | 0.6 | 23:28 | 4.8 |
| 28 | 5:11 | 1.5 | 11:27 | 5.8 | 6:08 | 0.7 | | |
| 29 | 0:18 | 4.3 | 5:41 | 1.9 | 12:04 | 5.7 | 19:07 | 0.8 |
| 30 | 1:26 | 3.8 | 6:16 | 2.5 | 12:53 | 5.6 | 20:24 | 0.9 |

The data can be obtained here:

The data covers one month.

The times are given using the '24-hour clock'.

The heights are measured from an arbitrary zero of depth. This is why there are some negative numbers. The unit is feet (USA).

The high and low tides are given in the order in which they occur. This is why there are some empty cells.

The high tides are in the blue cells.

## Modelling

The purpose of 'modelling' is to produce a mathematical entity that can be used to predict behaviour. In this cae, we are looking for a function of time that can be used to predict the tidal height at any time during the month. This can then be used to predict the depth of water anywhere in the harbour. This will enable predictions of time periods when shipping can move safely. This means clear of the sea bed and bridges etc.

There are two periodic aspects to the tides that need to be included in the model.

1. The twice daily cycle of high and low tides.

2. The variability of the heights of the high and low tides.

As a general comment about the first of these, the times of the first high tide each day are:

| Date | Time | Height | Time Difference |
|---|---|---|---|
| 1 | 1:15 | 4.0 | |
| 2 | 2:47 | 3.5 | 01:32:00 |
| 3 | 4:51 | 3.4 | 02:04:00 |
| 4 | 6:27 | 3.7 | 01:36:00 |
| 5 | 7:22 | 4.2 | 00:55:00 |
| 6 | 8:03 | 4.6 | 00:41:00 |
| 7 | 8:04 | 5.1 | 00:01:00 |
| 8 | 9:16 | 5.5 | 01:12:00 |
| 9 | 9:51 | 5.8 | 00:35:00 |
| 10 | 10:27 | 6.0 | 00:36:00 |
| 11 | 11:03 | 6.0 | 00:36:00 |
| 12 | 11:41 | 5.9 | 00:38:00 |
| 13 | 0:04 | 5.2 | 00:23:00 |
| 14 | 1:00 | 4.4 | 00:56:00 |
| 15 | 2:14 | 3.8 | 01:14:00 |
| 16 | 4:10 | 3.5 | 01:56:00 |
| 17 | 6:24 | 3.7 | 02:14:00 |
| 18 | 7:18 | 4.0 | 00:54:00 |
| 19 | 7:47 | 4.3 | 00:29:00 |
| 20 | 8:09 | 4.6 | 00:22:00 |
| 21 | 8:29 | 4.8 | 00:20:00 |
| 22 | 8:50 | 5.0 | 00:21:00 |
| 23 | 9:12 | 5.3 | 00:22:00 |
| 24 | 9:36 | 5.5 | 00:24:00 |
| 25 | 10:01 | 5.7 | 00:25:00 |
| 26 | 10:28 | 5.8 | 00:27:00 |
| 27 | 10:56 | 5.9 | 00:28:00 |
| 28 | 11:27 | 5.8 | 00:31:00 |
| 29 | 0:18 | 4.3 | 00:23:00 |
| 30 | 1:26 | 3.8 | 01:08:00 |

This sequence of high tides shows that high tide times appear to happen about an hour later each day. The period is neither 24 hours between this sequence of high tides nor 12 hours between successive high tides.

The first task is to try to model these timings. One of the problems is that the independent variable is time which is a non-decimal measuring system. One solution is to decimalise the times by using a formula such as:

$$t = 24 \times \text{day} + \text{hour} + \text{minutes} \div 60$$

At the first stage, we will just try to model the occasions on which the high and low tides occur. These first three days times are:

| Day | Time | Decimal Time | Tide |
|---|---|---|---|
| 1 | 1:15 | 25.25 | 1 |
| 1 | 6:56 | 30.93 | -1 |
| 1 | 13:38 | 37.63 | 1 |
| 1 | 20:42 | 44.70 | -1 |
| 2 | 2:47 | 50.78 | 1 |
| 2 | 7:45 | 55.75 | -1 |
| 2 | 14:44 | 62.73 | 1 |
| 2 | 22:18 | 70.30 | -1 |
| 3 | 4:51 | 76.85 | 1 |
| 3 | 9:15 | 81.25 | -1 |
| 3 | 16:04 | 88.07 | 1 |
| 3 | 23:42 | 95.70 | -1 |

As we are not yet trying to model the size of the tides, high tides are recorded as +1 and low tides as –1.

The table shows five cycles in a 88.07 – 25.25 = 62.82 hours.

The period is, therefore, 12.56.

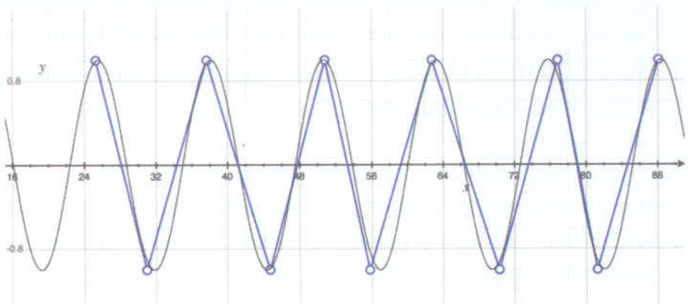
Using $\tau = \frac{2\pi}{n} \Rightarrow n = \frac{2\pi}{\tau}$ with the calculated period gives:

$$n = \frac{2\pi}{10.47} \approx 0.600$$

This suggests a modelling function of: $h(t)=\sin(0.5t+b)$.

The parameter $b$ must be chosen to synchronise the modelling function with the data. There are multiple values that work.

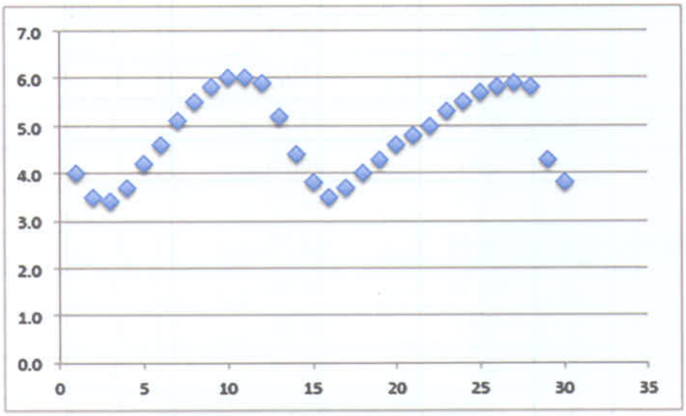
The following diagram shows the data points in blue (the lines joining them are unnecessary) and the modelling function $h(t)=\sin(0.5t+1.37)$.




We seem to have made a good start.

But what about the fact that the actual size of the tides also appears periodic?

This graph (produced using Excel) shows one of the high tides. It suggests some level of periodic behaviour.

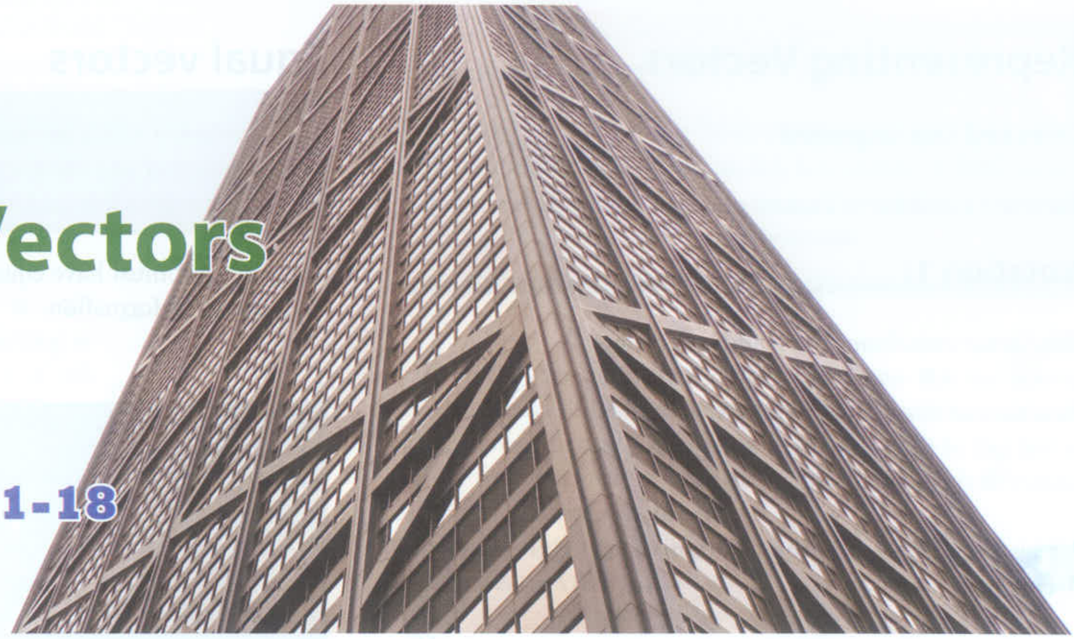



A full analysis would need to include both high tides and both low tides.

The period (if there is one!) appears to be close to a month. If this is true, why might this be?

After modelling the two parts of this problem, can you put the two parts together to produce a function that will predict the height of the water at time $t$?

# C.11 Vectors



## AHL 3.11-18

Modern structures such as our main picture are often made from supporting structures of steel wires and beams. These are frequently visible.

It is both the tension forces in these components and their direction that gives the building its strength. This dual feature (force and direction) defines a vector.

## Scalar and vector quantities

Numerical measurement scales are in widespread use. It is important to be able to distinguish between two distinct types of measurement scales, scalars and vectors.

### Scalar quantities

A scalar is a quantity that has magnitude (size) but no direction. For example, we measure the mass of objects using a variety of scales such as 'kilograms' and 'pounds'. These measures have magnitude in that more massive objects (such as the sun) have a larger numerical mass than small objects (such as this book). Giving the mass of this book does not, however, imply that this mass has a direction. This does **not** mean that scalar quantities must be positive. Signed scalar quantities, such as temperature as measured by the Celsius or Fahrenheit scales (which are commonly used) also exist.

### Vector quantities

Some measurements have both magnitude and direction. When we pull on a door handle, we exert what is known as a force. The force that we exert has both magnitude (we either pull hard or we pull gently) and direction (we open or close the door). Both the size of the pull and its direction are important in determining its effect. Such quantities are said to be vectors. Other examples of vectors are velocity, acceleration and displacement. The mathematics that will be developed in this section can be applied to problems involving any type of vector quantity.

### Exercise C.11.1

The following situations need to be described using an appropriate measure. Classify the measure as a scalar or a vector.

1. A classroom chair is moved from the front of the room to the back.
2. The balance in a bank account.
3. The electric current passing through an electric light tube.
4. A dog, out for a walk, is being restrained by a lead.
5. An aircraft starts its take-off run.
6. The wind conditions before a yacht race.
7. The amount of liquid in a jug.
8. The length of a car.

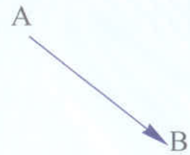
# Representing Vectors

## Directed line segment

There are a number of commonly used notations for vectors:

### Notation 1:

This vector runs from A to B and is depicted as $\overrightarrow{AB}$ or **AB** with the arrow giving the direction of the vector. Point A is known as the tail of the vector $\overrightarrow{AB}$ and point B is known as the head of vector $\overrightarrow{AB}$.

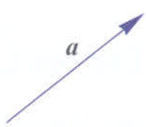



We also say the $\overrightarrow{AB}$ is the position vector of B relative to (from) A.

In the case where a vector starts at the origin (O), the vector running from O to another point C is simply called the position vector of C, $\overrightarrow{OC}$ or **OC**.


### Notation 2:

Rather than using two reference points, A and B, as in notation 1, we can also refer to a vector by making reference to a single letter attached to an arrow. In essence we are 'naming' the vector.




The vector $\boldsymbol{a}$ can be expressed in several ways. In text books they are often displayed in bold type, however, in written work, the following notations are generally used:

$$\boldsymbol{a} = \underset{\sim}{a} = \vec{a}$$

We will consider another vector notation later in this chapter.

## Magnitude of a vector

The magnitude or modulus of a vector is its length, which is the distance between its tail and head. We denote the magnitude of AB by |AB| (or more simply by AB).

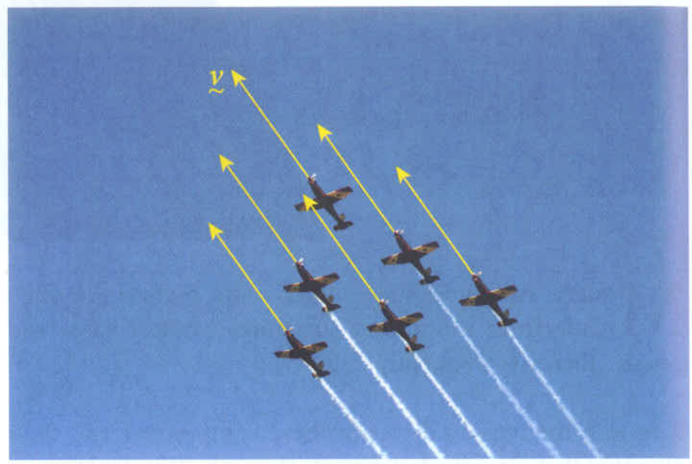
Similarly, if we are using vector notation 2, we may denote the magnitude of $\boldsymbol{a}$ by $|\boldsymbol{a}| = a$.

Note then that $|\boldsymbol{a}| \geq 0$.

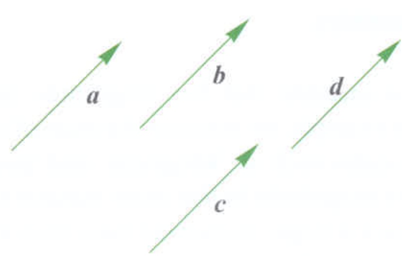
# Equal vectors

Two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are said to be equal if they have the same direction and the same magnitude, i.e. if they point in the same direction and $|\boldsymbol{a}| = |\boldsymbol{b}|$.

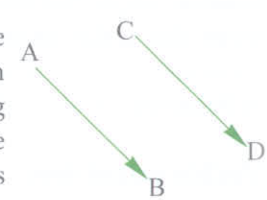
These aircraft must have equal velocity vectors if they are to maintain their formation.




Notice that if $\boldsymbol{a} = \boldsymbol{b}$, then vector $\boldsymbol{b}$ is a translation of vector $\boldsymbol{a}$. Using this notation, where there is no reference to a fixed point in space, we often use the term *free* vectors. That is, free vectors are vectors that have no specific position associated with them. In the diagram below, although the four vectors occupy a different space, they are all equal.

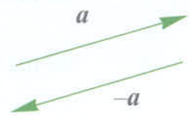



Note that we can also have that the vectors **AB** = **CD**, so that although they do not have the same starting point (or ending point) they are still equal because their magnitudes are equal and they have the same direction.



## Negative vectors

The negative of a vector $\boldsymbol{a}$, denoted by '$-\boldsymbol{a}$' is the vector $\boldsymbol{a}$ but pointing in the opposite direction to $\boldsymbol{a}$.




Similarly, the negative of **AB** is –**AB** or **BA**, because rather than starting at A and ending at B the negative of **AB** starts at B and ends at A.


Note that $|\boldsymbol{a}| = |-\boldsymbol{a}|$ and $|\mathbf{AB}| = |-\mathbf{AB}| = |\mathbf{BA}|$.

## Zero vector

The zero vector has zero magnitude, $|\mathbf{0}| = 0$ and has no definite direction. It is represented geometrically by joining a point onto itself. Note then that for any non-zero vector $\boldsymbol{a}$, $|\boldsymbol{a}| > 0$.

## Orientation and vectors

Vectors are useful when representing positions relative to some starting point. Consider:

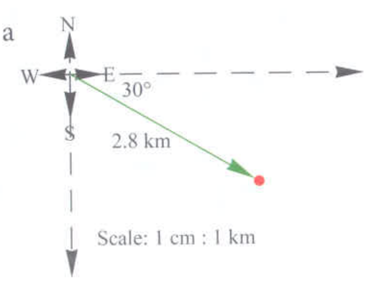
the position of a man who has walked 2.8 km across a field in a direction East 30° South or

a car moving at 20 km/h in a direction W 40° N for 2 hours.

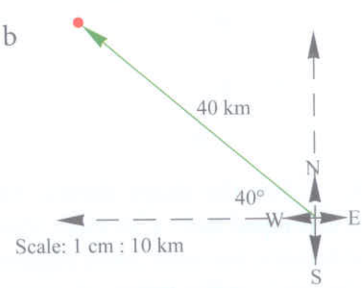
Each of these descriptions can be represented by a vector.

We start by setting up a set of axes and then we represent the above vectors showing the appropriate direction and magnitude. Representing the magnitude can be done using a scale drawing or labelling the length of the vector.

a



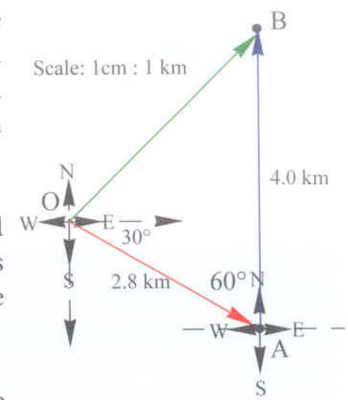

b




### Example C.11.1

Find the position of a bushwalker if, on the first part of her journey, she walks 2.8 km across a field in a direction East 30° South, and then continues for a further 4 km in a northerly direction.

We start by representing her journey using a vector diagram. The first part of her journey is represented by vector OA and the second partby AB. Note then that because her final position is at point B, her final position, relative to O, is given by the vector OB.




All that remains is to find the direction of OB and its magnitude. To do this we make use of trigonometry.

Finding $|\mathbf{OB}|$:Using the cosine rule we have:

$$OB^2 = OA^2 + AB^2 - 2(AB)(OA)\cos(60°)$$

$$= 2.8^2 + 4.0^2 - 2 \times 2.8 \times 4.0 \times 0.5$$

$$= 12.64$$

$$\therefore OB = 3.56$$

Next, we find the angle BOA:

$$AB^2 = OA^2 + OB^2 - 2(OA)(OB)\cos(\angle BOA)$$

$$4.0^2 = 2.8^2 + 12.64 - 2(2.8)(\sqrt{12.64})\cos(\angle BOA)$$

$$\therefore \cos(\angle BOA) = \frac{2.8^2 + 12.64 - 4.0^2}{2(2.8)(\sqrt{12.64})}$$

$$\angle BOA = \cos^{-1}(0.2250)$$

$$= 76°59'45''$$

$$\approx 77°$$

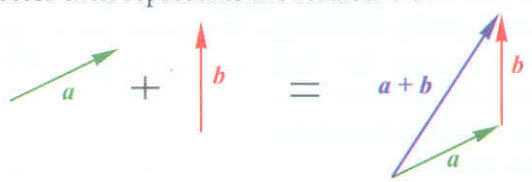
That is, the bushwalker is 3.56 km E 47° N from her starting point.

Although we will investigate the algebra of vectors in the next section, in Example C.11.1 we have already looked at adding two vectors informally. That is, the final vector **OB** was found by *joining* the vectors **OA** and **AB**. Writing this in vector form we have, $\mathbf{OB} = \mathbf{OA} + \mathbf{AB}$.

To add two vectors, ***a*** and ***b***, geometrically we

1. first draw ***a***,
2. draw vector ***b*** so that its tail meets the arrow end of vector ***a***,
3. draw a line segment from the tail of vector ***a*** to the arrow end of vector ***b***.

This vector then represents the result ***a*** + ***b***.

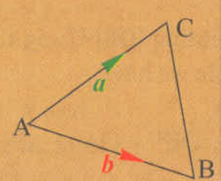



**Example C.11.2**

For the equilateral triangle shown, express in terms of *a* and *b*.

a CA b BC

c |**AB** + **BC**|




a **CA** = –**AC** = –***a***.

b To get from B to C we first get from B to A and then from A to C. That is, we 'join' the vectors **BA** and **AC**. In vector notation we have: **BC** = **BA** + **AC**

However, **AB** = ***b*** ⇒ **BA** = –**AB** = –***b***

∴**BC** = – ***b*** + ***a***

c **AB** + **BC** = **AC** = ***a*** ∴|**AB** + **BC**| = |***a***|

## Exercise C.11.2

1. Using a scale of 1 cm representing 10 units sketch the vectors that represent:

   a 30 km in a westerly direction.

   b 20 newtons applied in a NS direction.

   c 15 m/s N 60° E.

   d 45 km/h W 30° S.

2. The vector  represents a velocity of 20 $ms^{-1}$ due west. Represent the following vectors:

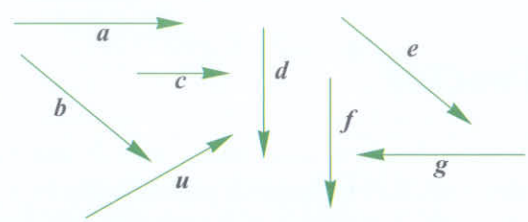
   a 20 $ms^{-1}$ due east

   b 40 $ms^{-1}$ due west

   c 60 $ms^{-1}$ due east

   d 40 $ms^{-1}$ due NE

3. State which of the vectors shown:




   a have the same magnitude.

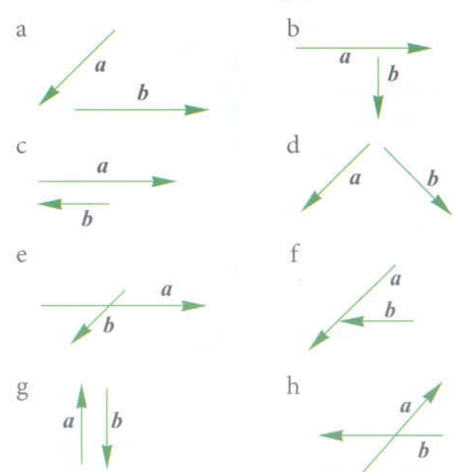
   b are in the same direction.

   c are in opposite directions.

   d are equal.

   e are parallel.

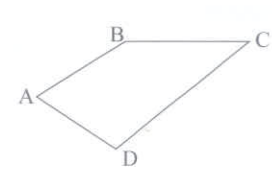
4. For each of the following pairs of vectors, find ***a*** + ***b***.




5. For the shape shown, find a single vector which is equal to:

   a **AB** + **BC**

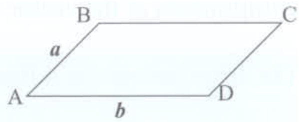
   b **AD** + **DB**



c **AC** + **CD**

d **BC** + **CD** + **DA** e **CD** + **DA** + **AB** + **BC**

6. Consider the parallelogram shown alongside. Which of the following statements are true?




a **AB** = **DC** b $|\boldsymbol{a}| = |\boldsymbol{b}|$

c **BC** = ***b*** d |**AC** + **CD**| = $|\boldsymbol{b}|$

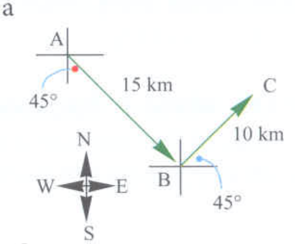
e **AD** = **CB**

7. For each of the following:

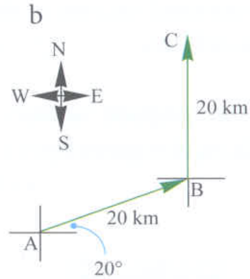
i complete the diagram by drawing the vector **AB**+**BC**.

ii find |**AB** + **BC**| .

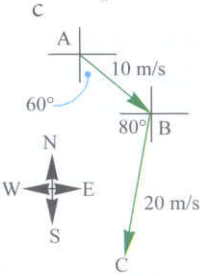
a




b




c




8. Two forces, one of 40 newtons acting in a northerly direction and one of 60 newtons acting in an easterly direction, are applied at a point A. Draw a vector diagram representing the forces. What is the resulting force at A?

9. Two trucks, on opposite sides of a river, are used to pull a barge along a straight river. They are connected to the barge at one point by ropes of equal length. The angle between the two ropes is 50°. Each truck is pulling with a force of 1500 newtons.

a Draw a vector diagram representing this situation.

b Find the magnitude and direction of the force acting on the barge.

10. An aircraft is flying at 240 km/h in a northerly direction when it encounters a 40 km/h wind from:

i the north. ii the north-east.

a Draw a vector diagram representing these situations.

b In each case, find the actual speed and direction of the aircraft.

11. Patrick walks for 200 m to point P due east of his cabin at point O, then 300 m due north where he reaches a vertical cliff, point Q. Patrick then climbs the 80 m cliff to point R.

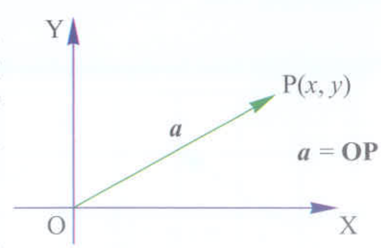
a Draw a vector diagram showing the vectors **OP**, **PQ** and **QR**.

b Find: i |**OQ**| ii |**OR**|

# Cartesian Representation of Vectors

## Representation in two dimensions

When describing vectors in two-dimensional space it is often helpful to make use of a rectangular Cartesian coordinate system.




As such, the position vector of the point P, **OP**, has the coordinates $(x, y)$.

The vector $\boldsymbol{a}$ can be expressed as a column vector $\begin{pmatrix} x \\ y \end{pmatrix}$.

That is:

> $\boldsymbol{a} = \begin{pmatrix} x \\ y \end{pmatrix}$ is the position vector OP where P has the coordinates $(x,y)$.

## Unit vector and base vector notation

We define the unit vector $i = \begin{pmatrix}1\\0\end{pmatrix}$

as the position vector of the point having coordinates (1, 0),

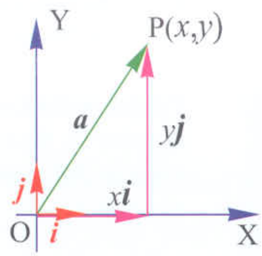
and the unit vector $j = \begin{pmatrix}0\\1\end{pmatrix}$

as the position vector of the point having coordinates (0, 1).

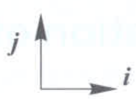
The term unit vector refers to the fact that the vector has a magnitude of one.

$$\boldsymbol{a} = \begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x\\0\end{pmatrix} + \begin{pmatrix}0\\y\end{pmatrix} = x\begin{pmatrix}1\\0\end{pmatrix} + y\begin{pmatrix}0\\1\end{pmatrix} = x\boldsymbol{i} + y\boldsymbol{j}$$

i.e. the position vector of any point can be expressed as the sum of two vectors, one parallel to the $x$-axis and one parallel to the $y$-axis.



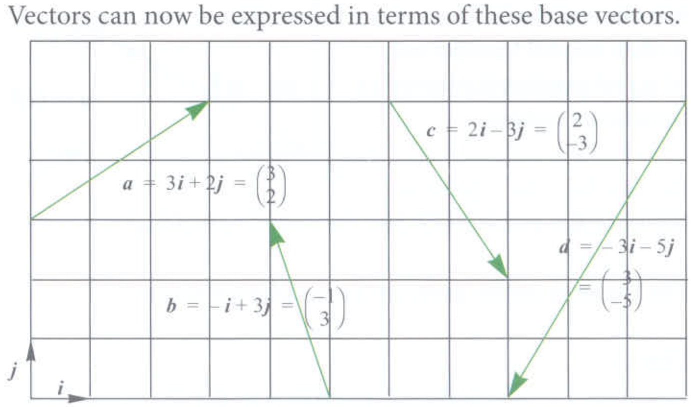

The unit vectors $\boldsymbol{i}$ and $\boldsymbol{j}$ are also known as the base vectors. If we confine ourselves to vectors that exist in the plane of this page, the most commonly used basis is:




where $|\boldsymbol{i}| = |\boldsymbol{j}| = 1$

Notice the definite direction of the base vectors, i.e. $\boldsymbol{i}$ points in the positive $x$-axis direction while $\boldsymbol{j}$ points in the positive $y$-axis direction.

Vectors can now be expressed in terms of these base vectors.



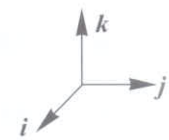

The vector $\boldsymbol{a}$ is 'three steps to the right and two steps up' and can be written in terms of the standard basis as $\boldsymbol{a} = 3\boldsymbol{i} + 2\boldsymbol{j}$.

The vector $\boldsymbol{b}$ is 'one step to the left and three steps up'. 'One step to the left' is in the opposite direction of the basis element $\boldsymbol{i}$ and is written $-\boldsymbol{i}$, giving the definition of the vector $\boldsymbol{b} = -\boldsymbol{i} + 3\boldsymbol{j}$. The vectors $-\boldsymbol{i}$ and $3\boldsymbol{j}$ are known as components of the vector $\boldsymbol{b}$.

The other definitions follow in a similar way.

## Representation in three dimensions

When vectors are represented in three-dimensional space, a third vector must be added to the basis, in this case it is a unit vector $\boldsymbol{k}$ and is such that the three unit vectors are mutually perpendicular as shown.




In addition, extra basis vectors can be added to generate higher dimensional vector spaces. These may not seem relevant to us, inhabiting as we do, a three dimensional space. However, it remains the case that it is possible to do calculations in higher dimensional spaces and these have produced many valuable results for applied mathematicians.

As was the case for vectors in two dimensions, we can represent vectors in three dimensions using column vectors as follows:

The position vector $\boldsymbol{a}$ = **OP** where P has coordinates $(x, y, z)$ is given by

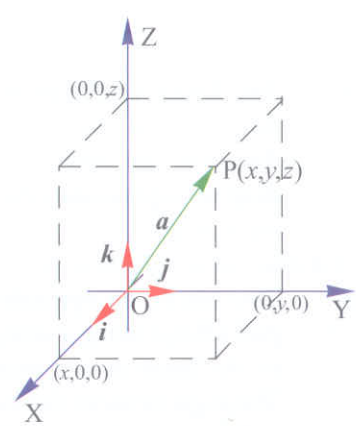
$$\boldsymbol{a} = \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}x\\0\\0\end{pmatrix} + \begin{pmatrix}0\\y\\0\end{pmatrix} + \begin{pmatrix}0\\0\\z\end{pmatrix} = x\begin{pmatrix}1\\0\\0\end{pmatrix} + y\begin{pmatrix}0\\1\\0\end{pmatrix} + z\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$= x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$$

Where this time the base vectors are:

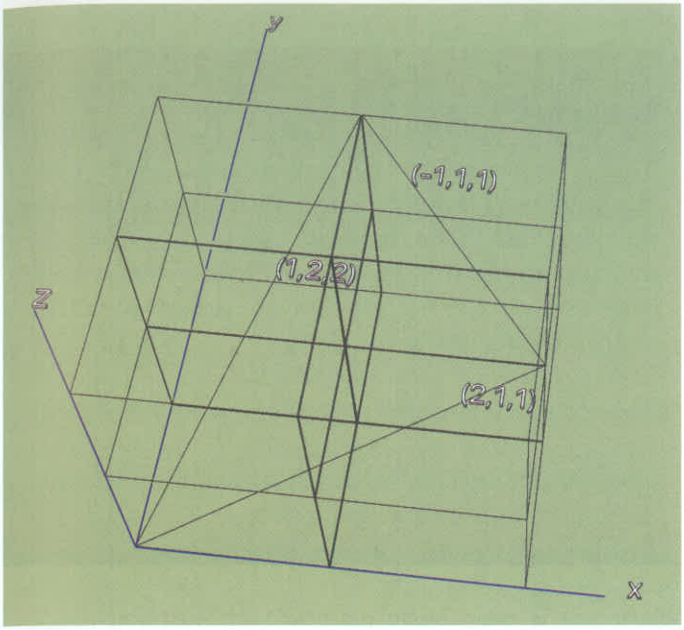
$$\boldsymbol{i} = \begin{pmatrix}1\\0\\0\end{pmatrix},\ \boldsymbol{j} = \begin{pmatrix}0\\1\\0\end{pmatrix} \text{ and}$$

$$\boldsymbol{k} = \begin{pmatrix}0\\0\\1\end{pmatrix}$$



Vectors in three dimensions can be difficult to visualise.

This diagram is a representation of the sum of vectors in three dimensions:




The diagram shows:

$$(2,1,1) + (-1,1,1) = (1,2,2)$$

The following QR code links to a 3 dimensional image of this calculation that you will be able to 'tumble' in order to get a better idea of the geometry of the situation.

3-d image. This file and the others like it in this chapter will need to be downloaded and viewed with an image viewer. Web browsers will not normally suffice.

# Vector operations

## Addition and subtraction

If $\boldsymbol{a} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = x_1\boldsymbol{i} + y_1\boldsymbol{j}$ and $\boldsymbol{b} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = x_2\boldsymbol{i} + y_2\boldsymbol{j}$ then:

$$\boldsymbol{a} \pm \boldsymbol{b} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \pm \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 \pm x_2 \\ y_1 \pm y_2 \end{pmatrix} = (x_1 \pm x_2)\boldsymbol{i} + (y_1 \pm y_2)\boldsymbol{j}$$

If $\boldsymbol{a} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = x_1\boldsymbol{i} + y_1\boldsymbol{j} + z_1\boldsymbol{k}$ and $\boldsymbol{b} = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = x_2\boldsymbol{i} + y_2\boldsymbol{j} + z_2\boldsymbol{k}$

then

$$\boldsymbol{a} \pm \boldsymbol{b} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \pm \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1 \pm x_2 \\ y_1 \pm y_2 \\ z_1 \pm z_2 \end{pmatrix}$$

$$= (x_1 \pm x_2)\boldsymbol{i} + (y_1 \pm y_2)\boldsymbol{j} + (z_1 \pm z_2)\boldsymbol{k}$$

## Scalar multiplication

If $\boldsymbol{a} = \begin{pmatrix} x \\ y \end{pmatrix} = x\boldsymbol{i} + y\boldsymbol{j}$

then $k\boldsymbol{a} = k\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} kx \\ ky \end{pmatrix} = kx\boldsymbol{i} + ky\boldsymbol{j}$ , $k \in \mathbb{R}$.

If $\boldsymbol{a} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ then:

$$k\boldsymbol{a} = k\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} kx \\ ky \\ kz \end{pmatrix} = kx\boldsymbol{i} + ky\boldsymbol{j} + kz\boldsymbol{k} \quad , k \in \mathbb{R}$$

**Example C.11.3**

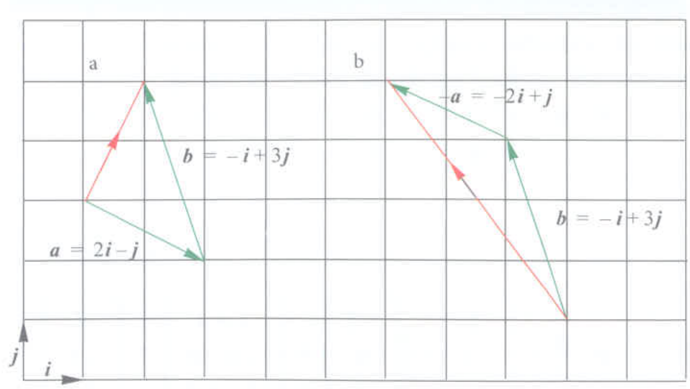
If $\boldsymbol{a} = 2\boldsymbol{i} - \boldsymbol{j}$ and $\boldsymbol{b} = -\boldsymbol{i} + 3\boldsymbol{j}$, find:

a $\boldsymbol{a} + \boldsymbol{b}$

b $\boldsymbol{b} - \boldsymbol{a}$

c $3\boldsymbol{b} - 2\boldsymbol{a}$

Vectors are added 'nose to tail':



a Vectors are added in much the same way as are algebraic terms. Only like terms can be added or subtracted, so that $\boldsymbol{a}+\boldsymbol{b} = (2\boldsymbol{i}-\boldsymbol{j})+(-\boldsymbol{i}+3\boldsymbol{j})$

$$= (2-1)\boldsymbol{i}+(-1+3)\boldsymbol{j}$$
$$= \boldsymbol{i}+2\boldsymbol{j}$$

b This problem is solved in a similar way:

$$\boldsymbol{b}-\boldsymbol{a} = (-\boldsymbol{i}+3\boldsymbol{j})-(2\boldsymbol{i}-\boldsymbol{j})$$
$$= (-1-2)\boldsymbol{i}+(3-(-1))\boldsymbol{j}$$
$$= -3\boldsymbol{i}+4\boldsymbol{j}$$

Note that we could also have expressed the sum as:

$$\boldsymbol{b}-\boldsymbol{a} = \boldsymbol{b}+(-\boldsymbol{a}) = (-\boldsymbol{i}+3\boldsymbol{j})+(-2\boldsymbol{i}+\boldsymbol{j})$$
$$= -3\boldsymbol{i}+4\boldsymbol{j}$$

(i.e. the negative of a vector is the same length as the original vector but points in the opposite direction.)

c Combining the properties of scalar multiplication with those of addition and subtraction we have:

$$3\boldsymbol{b}-2\boldsymbol{a} = 3(-\boldsymbol{i}+3\boldsymbol{j})-2(2\boldsymbol{i}-\boldsymbol{j})$$
$$= -3\boldsymbol{i}+9\boldsymbol{j}-4\boldsymbol{i}+2\boldsymbol{j}$$
$$= -7\boldsymbol{i}+11\boldsymbol{j}$$

## Example C.11.4

If $\boldsymbol{p} = \begin{pmatrix}3\\-1\\4\end{pmatrix}$ and $\boldsymbol{q} = \begin{pmatrix}-2\\0\\3\end{pmatrix}$, find:

a $\boldsymbol{p}+\boldsymbol{q}$ b $\boldsymbol{p}-\dfrac{\boldsymbol{q}}{2}$ c $\dfrac{3}{2}\boldsymbol{q}-\boldsymbol{p}$.

a $$\boldsymbol{p}+\boldsymbol{q} = \begin{pmatrix}3\\-1\\4\end{pmatrix}+\begin{pmatrix}-2\\0\\3\end{pmatrix} = \begin{pmatrix}3-2\\-1+0\\4+3\end{pmatrix} = \begin{pmatrix}1\\-1\\7\end{pmatrix}$$

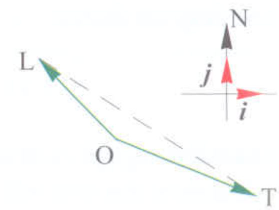
b $$\boldsymbol{p}-\frac{\boldsymbol{q}}{2} = \begin{pmatrix}3\\-1\\4\end{pmatrix}-\frac{1}{2}\begin{pmatrix}-2\\0\\3\end{pmatrix} = \begin{pmatrix}3+1\\-1-0\\4-1.5\end{pmatrix} = \begin{pmatrix}4\\-1\\2.5\end{pmatrix}$$

c $$\frac{3}{2}\boldsymbol{q}-\boldsymbol{p} = 1.5\begin{pmatrix}-2\\0\\3\end{pmatrix}-\begin{pmatrix}3\\-1\\4\end{pmatrix} = \begin{pmatrix}-3-3\\0+1\\4.5-4\end{pmatrix} = \begin{pmatrix}-6\\1\\0.5\end{pmatrix}$$

## Example C.11.5

A surveyor is standing at the top of a hill. Call this point 'the origin' (O). A lighthouse (L) is visible 4 km to the west and 3 km to the north. A town (T) is visible 5 km to the south and 2 km to the east. Using a vector basis in which $\boldsymbol{i}$ is a 1 km vector running east and $\boldsymbol{j}$ is a 1 km vector running north,

find the position vectors of the lighthouse, $\overrightarrow{OL}$ and the town $\overrightarrow{OT}$. Hence find the vector $\overrightarrow{LT}$ and the position of the town relative to the lighthouse.




The position vectors are:

Lighthouse $\overrightarrow{OL} = -4\boldsymbol{i}+3\boldsymbol{j}$ and

Town $\overrightarrow{OT} = 2\boldsymbol{i}-5\boldsymbol{j}$.

Then, to get from L to T we have $\overrightarrow{LT} = \overrightarrow{LO}+\overrightarrow{OT}$.

$$= -\overrightarrow{OL}+\overrightarrow{OT}$$
$$= -(-4\boldsymbol{i}+3\boldsymbol{j})+(2\boldsymbol{i}-5\boldsymbol{j})$$
$$= 4\boldsymbol{i}-3\boldsymbol{j}+2\boldsymbol{i}-5\boldsymbol{j}$$
$$= 6\boldsymbol{i}-8\boldsymbol{j}$$

This means that the town is 6 km east of the lighthouse and 8 km south.

## Exercise C.11.3

1. If $\boldsymbol{a} = \boldsymbol{i}+7\boldsymbol{j}-\boldsymbol{k}$ and $\boldsymbol{b} = 4\boldsymbol{i}+7\boldsymbol{j}+5\boldsymbol{k}$, find:

a $4\boldsymbol{a}$ b $3\boldsymbol{b}$

c $2\boldsymbol{a}-\boldsymbol{b}$ d $2(\boldsymbol{a}-\boldsymbol{b})$

2. The position vectors of A and B are $\overrightarrow{OA} = -3\boldsymbol{i} + 4\boldsymbol{j} - 2\boldsymbol{k}$ and $\overrightarrow{OB} = \boldsymbol{i} - 4\boldsymbol{j} - 3\boldsymbol{k}$. Find:

   a $\overrightarrow{AO}$ b $\overrightarrow{OA} - 5\overrightarrow{OB}$

   c $-5\overrightarrow{OA} + 3\overrightarrow{OB}$ d $3\overrightarrow{OA} + 6\overrightarrow{BO}$

3. If: $\boldsymbol{p} = \begin{pmatrix} -1 \\ -2 \\ 4 \end{pmatrix}$ and $\boldsymbol{q} = \begin{pmatrix} 6 \\ 1 \\ 2 \end{pmatrix}$. Find:

   a $\boldsymbol{p} + 2\boldsymbol{q}$ b $-3\boldsymbol{p} - 5\boldsymbol{q}$

   c $3\boldsymbol{p}$ d $2\boldsymbol{p} + 3\boldsymbol{q}$

4. Find the position vectors that join the origin to the points with coordinates A (2, –1) and B (–3, 2). Express your answers as column vectors. Hence find $\overrightarrow{AB}$.

5. A point on the Cartesian plane starts at the origin. The point then moves 4 units to the right, 5 units up, 6 units to the left and, finally 2 units down. Express these translations as a sum of four column vectors. Hence find the coordinates of the final position of the point.

6. Two vectors are defined as $\boldsymbol{a} = \boldsymbol{i} + \boldsymbol{j} + 4\boldsymbol{k}$ and $\boldsymbol{b} = -7\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$. Find:

   a $-6\boldsymbol{a} - 2\boldsymbol{b}$ b $-5\boldsymbol{a} + 2\boldsymbol{b}$

   c $4\boldsymbol{a} + 3\boldsymbol{b}$ d $-2(\boldsymbol{a} + 3\boldsymbol{b})$

7. If $\boldsymbol{x} = \begin{pmatrix} 4 \\ -4 \\ 2 \end{pmatrix}$ and $\boldsymbol{y} = \begin{pmatrix} 4 \\ 3 \\ 7 \end{pmatrix}$, find as column vectors.

   a $2\boldsymbol{x} + 3\boldsymbol{y}$ b $\boldsymbol{x} + 2\boldsymbol{y}$

   c $5\boldsymbol{x} - 6\boldsymbol{y}$ d $\boldsymbol{x} - 6\boldsymbol{y}$

8. Find the values of A and B if:

   $\mathrm{A}(7\boldsymbol{i} + 7\boldsymbol{j} + 4\boldsymbol{k}) - 3(3\boldsymbol{i} - \boldsymbol{j} + \mathrm{B}\boldsymbol{k}) = -37\boldsymbol{i} - 25\boldsymbol{j} + 5\boldsymbol{k}$

9. Two vectors are defined as:

   $\boldsymbol{a} = \begin{pmatrix} -3 \\ 1 \\ 4 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 6 \\ -6 \\ -5 \end{pmatrix}$.

   Find values of the scalars $X$ and $Y$ if $X\boldsymbol{a} + Y\boldsymbol{b}$ is equal to:

   a $\begin{pmatrix} -36 \\ 32 \\ 33 \end{pmatrix}$

   b $\begin{pmatrix} 30 \\ -22 \\ -31 \end{pmatrix}$

   c $\begin{pmatrix} -12 \\ 24 \\ 1 \end{pmatrix}$.

10. A submarine (which is considered the origin of the vector system) is 60 metres below the surface of the sea when it detects two surface ships. A destroyer (D) is 600 metres to the east and 800 metres to the south of the submarine. An aircraft carrier (A) is 1200 metres to the west and 300 metres to the south.

    a Define a suitable vector basis for this problem.

    b Using the submarine as the origin, state the position vectors of the destroyer and the aircraft carrier.

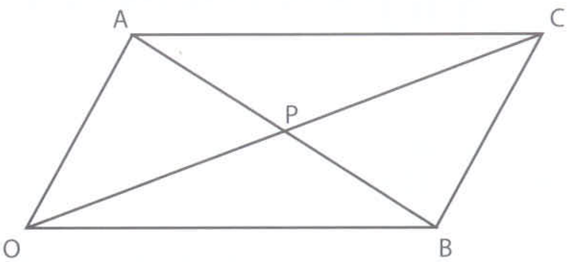
    c A helicopter pilot, based on the aircraft carrier, wants to make a supplies delivery to the destroyer. Find, in vector terms, the course along which the pilot should fly.

# Geometric proofs

Vector techniques can be used to prove some geometric proofs.

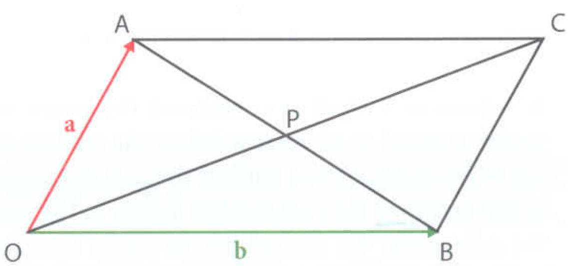
**Example C.11.6**

Prove that the diagonals of a parallelogram bisect one another.




The first step in constructing a vector proof is to set up and name some vectors that can be used to express the various parts of the diagram in vector terms.

If we make the following definitions: $\overrightarrow{OA}=\mathbf{a}$ and $\overrightarrow{OB}=\mathbf{b}$:




We can now use vector 'nose to tail' addition to express other parts of the diagram in terms of this 'basis'.

For example: $\overrightarrow{AB}=\overrightarrow{AO}+\overrightarrow{OB}$

$$=-\overrightarrow{OA}+\overrightarrow{OB}$$
$$=-\mathbf{a}+\mathbf{b}$$

In constructing a proof, it is important not to assume the truth of what you are trying to prove!

In this case, we do know that P is on AB, not that it is half way along it.

However, it must be the case that $\overrightarrow{AP}=k_1\times\overrightarrow{AB}$ where $k_1$ is some scalar factor.

It follows that: $\overrightarrow{AP}=k_1\times\overrightarrow{AB}$

$$=k_1\times(-\mathbf{a}+\mathbf{b})$$

Similarly: $\overrightarrow{OC}=\overrightarrow{OB}+\overrightarrow{BC}=\mathbf{b}+\mathbf{a}=\mathbf{a}+\mathbf{b}$ and P lies on OC.

Therefore there must be a second scalar $k_2$ such that:

$$\overrightarrow{OP}=k_2\times\overrightarrow{OC}$$
$$=k_2\times(\mathbf{a}+\mathbf{b})$$

Once again, we must avoid any assumptions about the two scalars being the same.

Looking at the triangle OAP: $\mathbf{a}=k_2\times(\mathbf{a}+\mathbf{b})-k_1\times(-\mathbf{a}+\mathbf{b})$

We now use the fact that **a** and **b** are vectors and that two vectors can only be equal if they have both the same magnitude and direction. This means that we must have the same multiple of both **a** and **b** on each side of the vector equation.

This leads to the technique known as 'equating coefficients'.

Firstly, we look at the vector **a**. On the left there is one of them and on the right we have $k_2 - (-k_1) = k_1 + k_2$.

This leads to the equation: $k_1 + k_2 = 1$

Equating the coefficients of **b** we get $0 = k_2 - k_1$

This pair of simultaneous equations is now solved:

$k_2 - k_1 = 0$ implies that $k_2 = k_1$.

This can be substituted into the other equation to give:

$k_1 + k_1 = 1$ so that $k_1 = k_2 = ½$ and we have proved that the diagonals bisect one another.

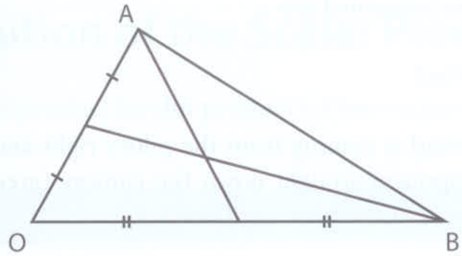
Not that the proof works for all quadrilaterals which have pairs of opposite sides parallel (square, rectangle & rhombus) but not for the kite.

The part of the proof in which this was used is: $\overrightarrow{BC}=\overrightarrow{OA}=\mathbf{a}$

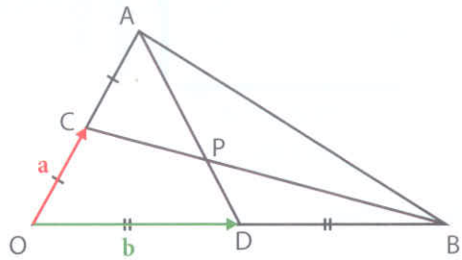
**Example C.11.7**

Prove that the medians of a triangle intersect one another two thirds of the way along their length.

The medians of a triangle join a vertex to the mid-point of the opposite side.




There are several ways of setting up the basis vectors. This one avoids fractions:




Following the method used in the previous example:

Using the fact that P lies on AD: $\overrightarrow{AP} = k_1 \times \overrightarrow{AD}$

and P lies on BC: $\overrightarrow{BP} = k_2 \times \overrightarrow{BC}$ .

Using the mid-point conditions: $\overrightarrow{OA} = 2\mathbf{a}$ and $\overrightarrow{OB} = 2\mathbf{b}$ .

Also: $\overrightarrow{AD} = \overrightarrow{AO} + \overrightarrow{OD}$ and $\overrightarrow{BC} = \overrightarrow{BO} + \overrightarrow{OC}$

$= -2\mathbf{a} + \mathbf{b}$ $\quad = -2\mathbf{b} + \mathbf{a}$

In triangle APB: 
$$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{AP} + \overrightarrow{PB} \\ &= \overrightarrow{AP} - \overrightarrow{BP} \\ &= k_1 \times \overrightarrow{AD} - k_2 \times \overrightarrow{BC} \\ &= k_1 \times (-2\mathbf{a} + \mathbf{b}) - k_2 \times (-2\mathbf{b} + \mathbf{a})\end{aligned}$$

but also: 
$$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{AO} + \overrightarrow{OB} \\ &= -2\mathbf{a} + 2\mathbf{b}\end{aligned}$$

This leads to the vector equation:

$$-2\mathbf{a} + 2\mathbf{b} = k_1 \times (-2\mathbf{a} + \mathbf{b}) - k_2 \times (-2\mathbf{b} + \mathbf{a})$$

Equating the coefficients of **a**: 
$$\begin{aligned}-2 &= -2k_1 - k_2 \\ 2 &= 2k_1 + k_2\end{aligned}$$

and of **b**: $2 = k_1 + 2k_2$

Subtracting twice the second equation from the first gives:

$$\begin{aligned}2 - 2\times 2 &= 2k_1 + k_2 - 2(k_1 + 2k_2) \\ -2 &= k_2 - 4k_2 \\ k_2 &= \frac{2}{3}\end{aligned}$$

and substituting this in the second equation gives:

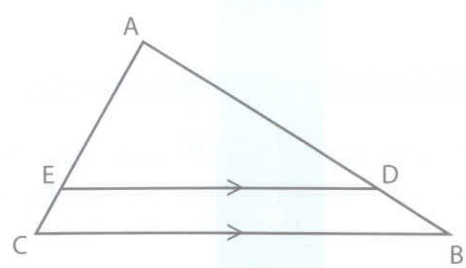
$$2 = k_1 + 2 \times \frac{2}{3}$$

$$\begin{aligned}k_1 &= 2 - \frac{4}{3} \\ &= \frac{2}{3}\end{aligned}$$

This is the result we were asked to prove. Note that this proof is not unique. Can you find a neater way of doing it?

## Exercise C.11.4

Prove, using vectors, that:

1. The line joining the mid points of two sides of a triangle is parallel to the third side.
2. The medians of a triangle are concurrent.
3. The altitudes of a triangle (lines joining each vertex that are perpendicular to the third side) are concurrent.
4. The space diagonals of a cuboid are concurrent and bisect one another.
5. The altitudes of a regular tetrahedron are concurrent.
6. The diagonals of a regular hexagon bisect one another.
7. The length of each side of a triangle is always less than the sum of the lengths of the other two sides.
8. 


AE:AC = AD:AB = DE:BC

# Application

## Crosswind Landing

This small aeroplane is landing at a short grass landing strip on a coral atoll.

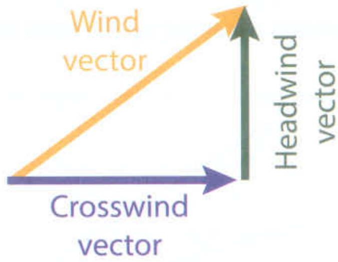
It appears that the aeroplane is heading almost straight for the camera.

Fortunately for the photographer, this is not so. The aircraft is appraching the landing strip 'crabwise' in order to offset the drift created by a cross wind.

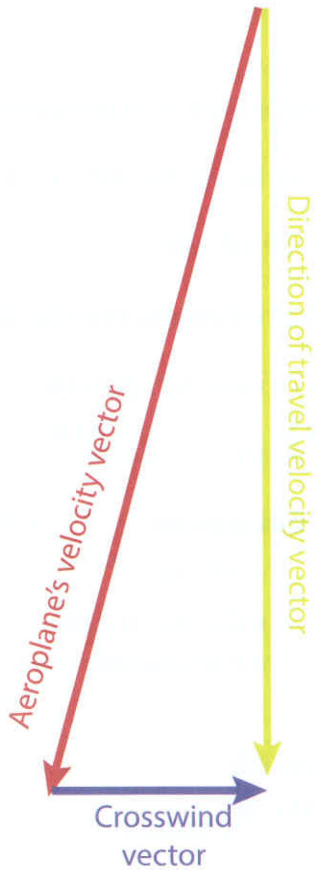

The vectors concerned are:

1. Wind

The crosswind is coming from the pilot's right and resolves into a component straight down the runway (green) and a component across the runway (blue).




2. The aeroplane



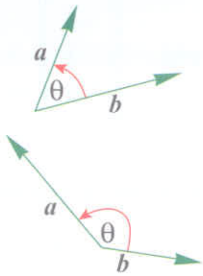

The crosswind component of the red vector balances the blue vector and the aeroplane travels straight down the runway. Without this, the aeroplane would drift off the runway line to the pilot's left during the approach. Just before touchdown, the pilot will straighten the aeroplane by using left rudder to 'yaw' it to the left. Some right aileron is also necessary to counteract the roll that happens during this 'de-crabbing'. Can you see why?

# Definition of the Scalar Product

The scalar product (or **dot product**) of two vectors is defined by:

$$\boldsymbol{a} \bullet \boldsymbol{b} = |\boldsymbol{a}|\,|\boldsymbol{b}| \cos\theta$$

where θ is the angle between the two vectors and may be an obtuse angle. The angle must be measured between the directions of the vectors. That is, the angle between the two vectors once they are joined tail to tail.



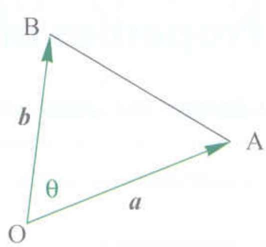

The three quantities on the right-hand side of the equation are all scalars and it is important to realise that, when the **scalar** product of two **vectors** is calculated, the result is a **scalar**.

**Example C.11.8**

Find the scalar product of the vectors $2\boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$.

Let $\boldsymbol{a} = 2\boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{b} = \boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$, then to determine the scalar product, $\boldsymbol{a} \bullet \boldsymbol{b}$, we need to find:

$|\boldsymbol{a}|$, $|\boldsymbol{b}|$ and $\cos\theta$, where θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$.




Finding:

$$|\boldsymbol{a}| = \sqrt{2^2 + (-3)^2 + 1^2} = \sqrt{14}.$$

$$|\boldsymbol{b}| = \sqrt{1^2 + 1^2 + (-1)^2} = \sqrt{3}.$$

Finding cosθ requires a little work. Relative to a common origin O, the points A(2, –3, 1) and B(1, 1, –1) have position vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.

Before making use of the cosine rule we need to determine the length of AB. Using the distance formula between two points in space, we have:

$$\begin{aligned} AB &= \sqrt{(1-2)^2 + (1-(-3))^2 + (-1-1)^2} \\ &= \sqrt{1+16+4} \\ &= \sqrt{21} \end{aligned}$$

Cosine rule:

$$AB^2 = OA^2 + OB^2 - 2 \cdot OA \cdot OB \cdot \cos\theta$$

$$(\sqrt{21})^2 = (\sqrt{14})^2 + (\sqrt{3})^2 - 2 \cdot \sqrt{14} \cdot \sqrt{3} \cdot \cos\theta$$

$$21 = 14 + 3 - 2\sqrt{42}\cos\theta$$

$$\therefore \cos\theta = -\frac{2}{\sqrt{42}}$$

Next, from the definition of the scalar product:

$\boldsymbol{a} \bullet \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\cos\theta$, we have

$$\boldsymbol{a} \bullet \boldsymbol{b} = \sqrt{14} \times \sqrt{3} \times -\frac{2}{\sqrt{42}} = -2$$

The solution to Example C.11.8 was rather lengthy. However, we now look at the scalar product from a slightly different viewpoint.

First consider the dot product $\boldsymbol{i} \bullet \boldsymbol{i}$:

Using the definition, we have that

$$\boldsymbol{i} \bullet \boldsymbol{i} = |\boldsymbol{i}||\boldsymbol{i}|\cos 0 = 1 \times 1 \times 1 = 1$$

(the angle between the vectors $\boldsymbol{i}$ and $\boldsymbol{i}$ is 0 and so cosθ = cos0 = 1).

Next consider the product $\boldsymbol{i} \bullet \boldsymbol{j}$:

Using the definition, we have that:

$$\boldsymbol{i} \bullet \boldsymbol{j} = |\boldsymbol{i}||\boldsymbol{j}|\cos 90 = 1 \times 1 \times 0 = 0$$

(the angle between the vectors $\boldsymbol{i}$ and $\boldsymbol{j}$ is 90° and so cosθ = cos90° = 0).

Similarly, we end up with the following results for all possible combinations of the $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ vectors:

$$\boldsymbol{i} \bullet \boldsymbol{i} = \boldsymbol{j} \bullet \boldsymbol{j} = \boldsymbol{k} \bullet \boldsymbol{k} = 1$$

and

$$\boldsymbol{i} \cdot \boldsymbol{j} = \boldsymbol{i} \cdot \boldsymbol{k} = \boldsymbol{j} \cdot \boldsymbol{k} = \boldsymbol{j} \cdot \boldsymbol{i} = \boldsymbol{k} \cdot \boldsymbol{i} = \boldsymbol{k} \cdot \boldsymbol{j} = 0$$

Armed with these results we can now work out the scalar product of the vectors $\boldsymbol{a} = x_1\boldsymbol{i} + y_1\boldsymbol{j} + z_1\boldsymbol{k}$ and $\boldsymbol{b} = x_2\boldsymbol{i} + y_2\boldsymbol{j} + z_2\boldsymbol{k}$ as follows:

$$\begin{aligned} a \bullet b &= (x_1\boldsymbol{i}+y_1\boldsymbol{j}+z_1\boldsymbol{k}) \bullet (x_2\boldsymbol{i}+y_2\boldsymbol{j}+z_2\boldsymbol{k}) \\ &= x_1x_2(\boldsymbol{i}\bullet\boldsymbol{i})+x_1y_2(\boldsymbol{i}\cdot\boldsymbol{j})+x_1z_2(\boldsymbol{i}\cdot\boldsymbol{k}) \\ &\quad +y_1x_2(\boldsymbol{j}\cdot\boldsymbol{i})+y_1y_2(\boldsymbol{j}\bullet\boldsymbol{j})+y_1z_2(\boldsymbol{j}\cdot\boldsymbol{k}) \\ &\quad +z_1x_2(\boldsymbol{k}\cdot\boldsymbol{i})+z_1y_2(\boldsymbol{k}\cdot\boldsymbol{j})+z_1z_2(\boldsymbol{k}\bullet\boldsymbol{k}) \end{aligned}$$

$$a \bullet b = x_1x_2+y_1y_2+z_1z_2$$

That is, if:

$$\boldsymbol{a} = x_1\boldsymbol{i}+y_1\boldsymbol{j}+z_1\boldsymbol{k} \text{ and } \boldsymbol{b} = x_2\boldsymbol{i}+y_2\boldsymbol{j}+z_2\boldsymbol{k}$$

$$\text{then } a \bullet b = x_1x_2+y_1y_2+z_1z_2$$

Using this result with the vectors of Example 4.2.1, $2\boldsymbol{i}-3\boldsymbol{j}+\boldsymbol{k}$ and $\boldsymbol{i}+\boldsymbol{j}-\boldsymbol{k}$ we have:

$$\begin{aligned} (2\boldsymbol{i}-3\boldsymbol{j}+\boldsymbol{k})\bullet(\boldsymbol{i}+\boldsymbol{j}-\boldsymbol{k}) &= 2\times1+(-3)\times1+1\times(-1) \\ &= 2-3-1 \\ &= -2 \end{aligned}$$

This is a much faster process!

However, the most usual use of scalar product is to calculate the angle between vectors using a rearrangement of the definition of scalar product:

$$\cos\theta = \frac{\boldsymbol{a}\bullet\boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|}$$

**Example C.11.9**

For the following pairs of vectors, find their magnitudes and scalar products. Hence find the angles between the vectors, correct to the nearest degree.

a $\quad -\boldsymbol{i}+3\boldsymbol{j}$ and $-\boldsymbol{i}+2\boldsymbol{j}$

b $\quad \begin{pmatrix}0\\-5\\4\end{pmatrix}$ and $\begin{pmatrix}-5\\-1\\-3\end{pmatrix}$.

a In using the scalar product, it is necessary to calculate the magnitudes of the vectors.

$|-i+3j|=\sqrt{(-1)^2+3^2}=\sqrt{10}$ and

$|-i+2j|=\sqrt{(-1)^2+2^2}=\sqrt{5}$

Next, calculate the scalar product: $-\boldsymbol{i}+3\boldsymbol{j} \quad -\boldsymbol{i}+2\boldsymbol{j}$

$$(-\boldsymbol{i}+3\boldsymbol{j})\bullet(-\boldsymbol{i}+2\boldsymbol{j}) = -1\times-1+3\times2 = 7$$

Finally, the angle is: $\cos\theta = \frac{\boldsymbol{a}\bullet\boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|} = \frac{7}{\sqrt{10}\times\sqrt{5}} \Rightarrow \theta \approx 8°$

b $$\left|\begin{pmatrix}0\\-5\\4\end{pmatrix}\right| = \sqrt{0^2+(-5)^2+4^2} = \sqrt{41}$$

and $$\left|\begin{pmatrix}-5\\-1\\-3\end{pmatrix}\right| = \sqrt{(-5)^2+(-1)^2+(-3)^2} = \sqrt{35}$$

Next, the scalar product:

$$\begin{pmatrix}0\\-5\\4\end{pmatrix}\bullet\begin{pmatrix}-5\\-1\\-3\end{pmatrix} = 0\times(-5)+(-5)\times(-1)+4\times(-3) = -7$$

Finally, the angle can be calculated:

$$\cos\theta = \frac{\boldsymbol{a}\bullet\boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|} = \frac{-7}{\sqrt{41}\times\sqrt{35}} \Rightarrow \theta \approx 101°$$

The use of cosine means that obtuse angles between vectors (which occur when the scalar product is negative) are calculated correctly when using the inverse cosine function on a calculator.

## Properties of the Scalar Product

**Closure** The scalar product of two vectors is a scalar (i.e. the result is not a vector). The operation is not closed and so closure does not apply.

**Commutative**

Now, $\boldsymbol{a}\bullet\boldsymbol{b} = |a||b|\cos\theta = |b||a|\cos\theta = \boldsymbol{b}\bullet\boldsymbol{a}$

That is, $\boldsymbol{a}\bullet\boldsymbol{b} = \boldsymbol{b}\bullet\boldsymbol{a}$.

Therefore the operation of scalar product is commutative.

**Associative** If the associative property were to hold it would take on the form $(\boldsymbol{a}\bullet\boldsymbol{b})\bullet\boldsymbol{c} = \boldsymbol{a}\bullet(\boldsymbol{b}\bullet\boldsymbol{c})$. However, $\boldsymbol{a}\bullet\boldsymbol{b}$ is a real number and therefore the operation $(\boldsymbol{a}\bullet\boldsymbol{b})\bullet\boldsymbol{c}$ has no meaning (you cannot 'dot' a scalar with a vector).

**Distributive** The **scalar product is distributive** (over addition).

We leave the proof of this result as an exercise - it was assumed in the discussion on the previous page.

**Identity** As the operation of scalar product is not closed, an identity cannot exist.

**Inverse** As the operation of scalar product is not closed, an inverse cannot exist.

Note that although the scalar product is non-associative, the following 'associative rule' holds for the scalar product:

If $k \in \mathbb{R}$, then, $\boldsymbol{a} \bullet (k\boldsymbol{b}) = k(\boldsymbol{a} \bullet \boldsymbol{b})$

## Special cases of the scalar product

### Perpendicular vectors

If the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular then:

$$\boldsymbol{a} \bullet \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\cos\frac{\pi}{2} = 0\,.$$

(Note: We are assuming that $\boldsymbol{a}$ and $\boldsymbol{b}$ are non-zero vectors.)

### Zero vector

For any vector $\boldsymbol{a}$, $\boldsymbol{a} \bullet \boldsymbol{0}$: $\boldsymbol{a} \bullet \boldsymbol{0} = |\boldsymbol{a}||\boldsymbol{0}|\cos\theta = 0$

### Parallel vectors

If vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel then, $\boldsymbol{a} \bullet \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\cos 0 = |\boldsymbol{a}||\boldsymbol{b}|$

If $\boldsymbol{a}$ and $\boldsymbol{b}$ are antiparallel then, $\boldsymbol{a} \bullet \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\cos\pi = -|\boldsymbol{a}||\boldsymbol{b}|$ .

(Note: We are assuming that $\boldsymbol{a}$ and $\boldsymbol{b}$ are non-zero vectors.)

Combining the results of 1 and 2 above, we have the important observation:

If $\boldsymbol{a} \bullet \boldsymbol{b} = 0$ then either:

1. $\boldsymbol{a}$ and/or $\boldsymbol{b}$ are both the zero vector, $\boldsymbol{0}$.

Or

2. $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular with neither $\boldsymbol{a}$ nor $\boldsymbol{b}$ being the zero vector.

Notice how this result differs from the standard Null Factor Law when dealing with real numbers, where given $ab = 0$ then $a$ or $b$ or both are zero! That is, the cancellation property that holds for real numbers does not hold for vectors.

A nice application using the perpendicular property above can be seen in the next example.

**Example C.11.10**

Three towns are joined by straight roads. Oakham is the state capital and is considered as the 'origin'. Axthorp is 3 km east and 9 km north of Oakham and Bostock is 5 km east and 5 km south of Axthorp.

Considering $\boldsymbol{i}$ as a 1 km vector pointing east and $\boldsymbol{j}$ a 1 km vector pointing north:

a Find the position vector of Axthorp relative to Oakham.

b Find the position vector of Bostock relative to Oakham.

A bus stop (S) is situated two thirds of the way along the road from Oakham to Axthorp.

c Find the vectors $\overline{\text{OS}}$ and $\overline{\text{BS}}$.

d Prove that the bus stop is the closest point to Bostock on the Oakham to Axthorp road.

a Axthorp is 3 km east and 9 km north of Oakham so $\overrightarrow{OA} = 3\boldsymbol{i} + 9\boldsymbol{j}$

b $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB} = 3\boldsymbol{i} + 9\boldsymbol{j} + 5\boldsymbol{i} - 5\boldsymbol{j} = 8\boldsymbol{i} + 4\boldsymbol{j}$

c $\overrightarrow{OS} = \frac{2}{3}(\overrightarrow{OA}) = \frac{2}{3}(3\boldsymbol{i} + 9\boldsymbol{j}) = 2\boldsymbol{i} + 6\boldsymbol{j}$

$\overrightarrow{BS} = \overrightarrow{BO} + \overrightarrow{OS} = -(8\boldsymbol{i} + 4\boldsymbol{j}) + 2\boldsymbol{i} + 6\boldsymbol{j} = -6\boldsymbol{i} + 2\boldsymbol{j}$

d The next step is to calculate the angle between AD and $\overrightarrow{BS}$ by calculating the scalar product of the two vectors:

$$\overrightarrow{OS} \bullet \overrightarrow{BS} = (2\boldsymbol{i} + 6\boldsymbol{j}) \bullet (-6\boldsymbol{i} + 2\boldsymbol{j}) = 2 \times (-6) + 6 \times 2 = 0$$

This means that $\overrightarrow{OS}$ and $\overrightarrow{BS}$ are at right angles to each other. It follows that the bus stop is the closest point to Bostock on the Oakham to Axthorp road.

### Example C.11.11

Find the value(s) of $m$ for which the vectors $2m\boldsymbol{i} + m\boldsymbol{j} + 8\boldsymbol{k}$ and $\boldsymbol{i} + 3m\boldsymbol{j} - \boldsymbol{k}$ are perpendicular.

As the two vectors are perpendicular, then:

$$(2m\boldsymbol{i} + m\boldsymbol{j} + 8\boldsymbol{k}) \bullet (\boldsymbol{i} + 3m\boldsymbol{j} - \boldsymbol{k}) = 0$$

$$\Rightarrow 2m + 3m^2 - 8 = 0$$

$$\Leftrightarrow 3m^2 + 2m - 8 = 0$$

$$\Leftrightarrow (3m-4)(m+2) = 0$$

$$\Leftrightarrow m = \frac{4}{3} \text{ or } m = -2$$

### Example C.11.12

Find a vector perpendicular to $\boldsymbol{u} = 4\boldsymbol{i} - 3\boldsymbol{j}$.

Let the vector perpendicular to $\boldsymbol{u} = 4\boldsymbol{i} - 3\boldsymbol{j}$ be $\boldsymbol{v} = x\boldsymbol{i} + y\boldsymbol{j}$.

Then, as $\boldsymbol{u} \perp \boldsymbol{v} \Rightarrow \boldsymbol{u} \bullet \boldsymbol{v} = 0$ so that $(4\boldsymbol{i} - 3\boldsymbol{j}) \bullet (x\boldsymbol{i} + y\boldsymbol{j}) = 0$

$$\therefore 4x - 3y = 0 \quad - (1)$$

Unfortunately, at this stage we only have one equation for two unknowns! We need to obtain a second equation from somewhere. To do this we recognise the fact that if $\boldsymbol{v}$ is perpendicular to $\boldsymbol{u}$, then so too will the unit vector, $\hat{v}$, be perpendicular to $\boldsymbol{u}$.

Then, as $|\hat{v}| = 1 \Rightarrow \sqrt{x^2 + y^2} = 1 \therefore x^2 + y^2 = 1 \quad - (2)$

From (1) we have that $y = \frac{4}{3}x \quad - (3)$

Substituting (3) into (2) we have:

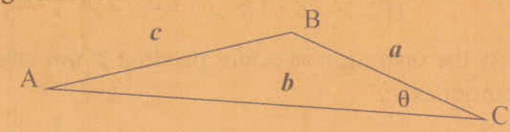
$$x^2 + \left(\frac{4}{3}x\right)^2 = 1 \Leftrightarrow 25x^2 = 9 \Leftrightarrow x = \pm\frac{3}{5}$$

Substituting into (3) we have: $y = \pm\frac{4}{5}$

Therefore, both $\boldsymbol{v} = \frac{3}{5}\boldsymbol{i} + \frac{4}{5}\boldsymbol{j}$ and $\boldsymbol{v} = -\left(\frac{3}{5}\boldsymbol{i} + \frac{4}{5}\boldsymbol{j}\right)$ are perpendicular to $\boldsymbol{u}$.

### Example C.11.13

Use a vector method to derive the cosine rule for the triangle shown.




From the triangle rule for vector addition we have $\boldsymbol{a} + \boldsymbol{c} = \boldsymbol{b} \Leftrightarrow \boldsymbol{c} = \boldsymbol{b} - \boldsymbol{a}$.

Now, using the scalar product we have:

$$\boldsymbol{c} \bullet \boldsymbol{c} = (\boldsymbol{b} - \boldsymbol{a}) \bullet (\boldsymbol{b} - \boldsymbol{a})$$

$$= \boldsymbol{b} \bullet \boldsymbol{b} - \boldsymbol{b} \bullet \boldsymbol{a} - \boldsymbol{a} \bullet \boldsymbol{b} + \boldsymbol{a} \bullet \boldsymbol{a}$$

$$= |\boldsymbol{b}|^2 - 2\boldsymbol{a} \bullet \boldsymbol{b} + |\boldsymbol{a}|^2$$

$$\therefore |\boldsymbol{c}|^2 = |\boldsymbol{b}|^2 + |\boldsymbol{a}|^2 - 2|\boldsymbol{a}||\boldsymbol{b}|\cos\theta$$

### Example C.11.14

Find a vector perpendicular to both $\boldsymbol{a} = 2\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$ and $\boldsymbol{b} = \boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k}$.

Let the vector $\boldsymbol{c} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ be perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$.

Then, we have that $\boldsymbol{a} \bullet \boldsymbol{c} = 0$ and $\boldsymbol{b} \bullet \boldsymbol{c} = 0$.

From $\boldsymbol{a} \bullet \boldsymbol{c} = 0$ we obtain:

$$(2\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}) \bullet (x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}) = 2x + y - z = 0 \quad - (1)$$

From $\boldsymbol{b} \bullet \boldsymbol{c} = 0$ we obtain:

$$(\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k}) \bullet (x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}) = x + 3y + z = 0 \quad - (2)$$

In order to solve for the three unknowns we need one more equation. We note that if $\boldsymbol{c}$ is perpendicular to $\boldsymbol{a}$ and $\boldsymbol{b}$ then so too will the unit vector, $\hat{c}$. So, without any loss in generality, we can assume that $\boldsymbol{c}$ is a unit vector. This will provide a third equation.

As we are assuming that $\boldsymbol{c}$ is a unit vector, we have:

$$|\boldsymbol{c}| = 1 \therefore x^2 + y^2 + x^2 = 1 \quad - (3)$$

We can now solve for $x$, $y$ and $z$:

(1) + (2): $3x + 4y = 0$ – (4)

2×(1) – (2): $5y + 3z = 0$ – (5)

Substituting (4) and (5) into (3): $\left(-\frac{4}{3}y\right)^2 + y^2 + \left(-\frac{5}{3}y\right)^2 = 1$

$$\Leftrightarrow 16y^2 + 9y^2 + 25y^2 = 9$$
$$\Leftrightarrow 50y^2 = 9$$
$$\Leftrightarrow y = \pm\frac{3}{5\sqrt{2}}$$
$$\therefore y = \pm\frac{3\sqrt{2}}{10}$$

Substituting into (4) and (5) we have $x = -\frac{4}{3} \times \pm\frac{3\sqrt{2}}{10} = \pm\frac{2\sqrt{2}}{5}$

and $z = -\frac{5}{3} \times \pm\frac{3\sqrt{2}}{10} = \pm\frac{\sqrt{2}}{2}$.

Therefore, $\pm\frac{2\sqrt{2}}{5}\boldsymbol{i} \pm \frac{3\sqrt{2}}{10}\boldsymbol{j} \pm \frac{\sqrt{2}}{2}\boldsymbol{k}$ or $\pm\left(\frac{2\sqrt{2}}{5}\boldsymbol{i} - \frac{3\sqrt{2}}{10}\boldsymbol{j} + \frac{\sqrt{2}}{2}\boldsymbol{k}\right)$

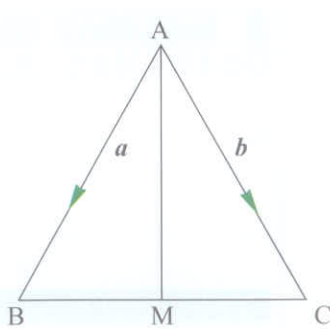
are two vectors perpendicular to $\boldsymbol{a}$ and $\boldsymbol{b}$. Of course, any multiple of this vector will also be perpendicular to $\boldsymbol{a}$ and $\boldsymbol{b}$.

As we have a seen in these examples, the scalar product is a very powerful tool when proving theorems in geometry. We now look at another theorem that is otherwise lengthy to prove by standard means.

**Example C.11.15**

Prove that the median to the base of an isosceles triangle is perpendicular to the base.

Consider the triangle ABC as shown, where M is the midpoint of the base $\overline{BC}$. Next, let $\boldsymbol{a} = \mathbf{AB}$ and $\boldsymbol{b} = \mathbf{AC}$. We then wish to show that $\mathbf{AM} \perp \mathbf{BC}$ (or $\mathbf{AM} \bullet \mathbf{BC} = 0$).




Now, $\mathbf{AM} = \mathbf{AB} + \mathbf{BM} = \mathbf{AB} + \frac{1}{2}\mathbf{BC}$

$$= \boldsymbol{a} + \frac{1}{2}(\boldsymbol{b} - \boldsymbol{a})$$
$$= \frac{1}{2}(\boldsymbol{a} + \boldsymbol{b})$$

Therefore, $\mathbf{AM} \bullet \mathbf{BC} = \frac{1}{2}(\boldsymbol{a} + \boldsymbol{b}) \bullet (\boldsymbol{b} - \boldsymbol{a})$

$$= \frac{1}{2}(\boldsymbol{a} \bullet \boldsymbol{b} - \boldsymbol{a} \bullet \boldsymbol{a} + \boldsymbol{b} \bullet \boldsymbol{b} - \boldsymbol{b} \bullet \boldsymbol{a})$$

$= \frac{1}{2}(-|\boldsymbol{a}|^2 + |\boldsymbol{b}|^2)$ (because $\boldsymbol{a} \bullet \boldsymbol{b} = \boldsymbol{b} \bullet \boldsymbol{a}$)

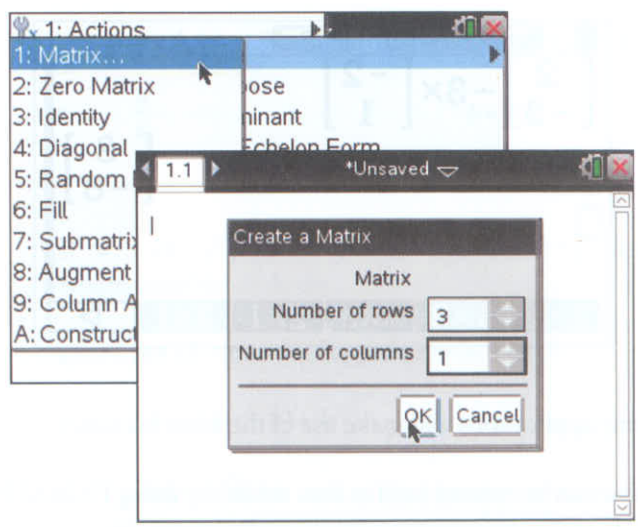
$= 0$ (because $|\boldsymbol{a}| = |\boldsymbol{b}|$)

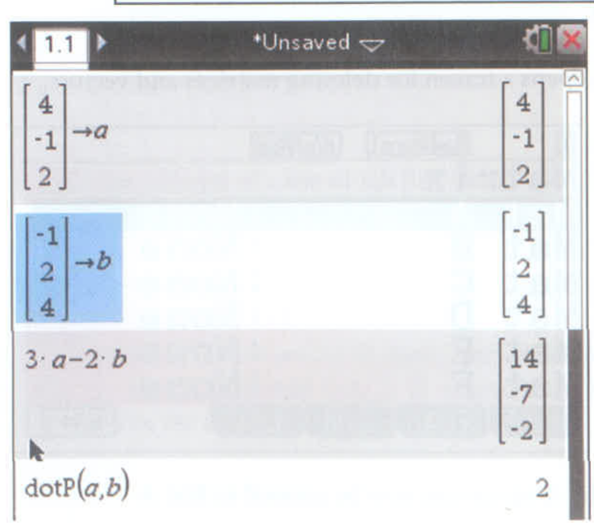
Therefore:

As $\mathbf{AM} \neq \mathbf{0}$ and $\mathbf{BC} \neq \mathbf{0}$, then $\mathbf{AM} \bullet \mathbf{BC} = 0 \Rightarrow \mathbf{AM} \perp \mathbf{BC}$

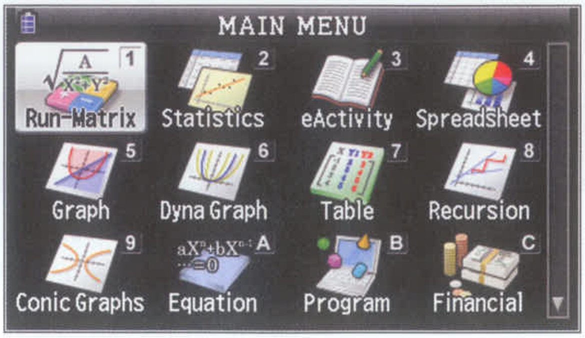
i.e. the median is perpendicular to the base.

Most graphic calculators can perform vector calculations. You should know how to do the basic procedures such as entering and saving vectors.



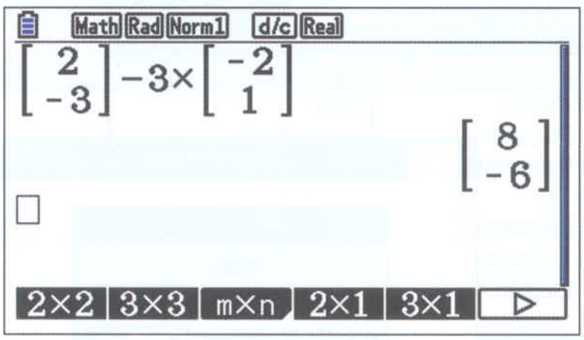
If using Casio, select Module 1.




Then the calculation continues:

Vectors can be entered as needed and arithmetic performed on them by pressing F4-MATH and F1-MAT/VCT. This provides a screen from which common vector (and matrix) layouts can be accessed and basic operations performed.

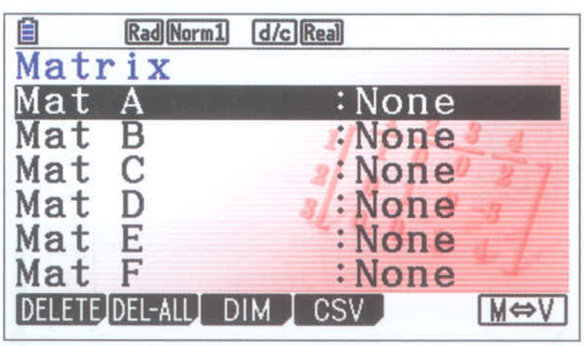
If using 2 by 1 vectors, a blank vector of the right size can be found by pressing F4. The values can now be entered from the keyboard.



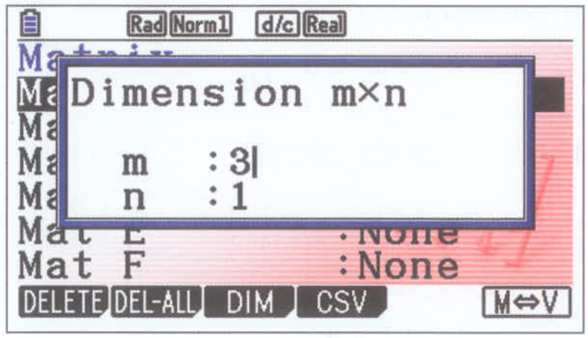

Many applications will make use of the same vectors.

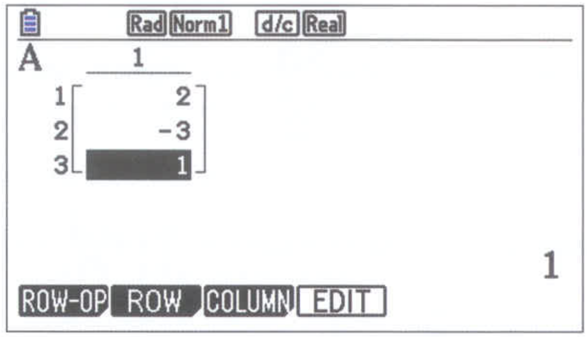
These can be entered (still in Run mode) by using F3-MATH/VCT.

This opens a screen for defining matrices and vectors.




A 3 by 1 vector can now be entered as Mat A

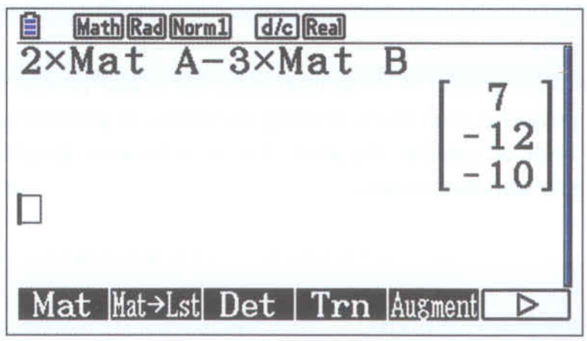


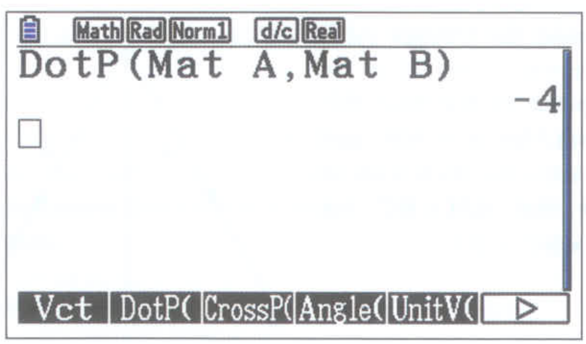

If a second vector is stored $\begin{bmatrix}-1\\2\\4\end{bmatrix}$,

both can be accessed repeatedly to perform calculations.

The vector A is accessed by pressing OPTN, F2, F1 followed by ALPHA A to name the vector.

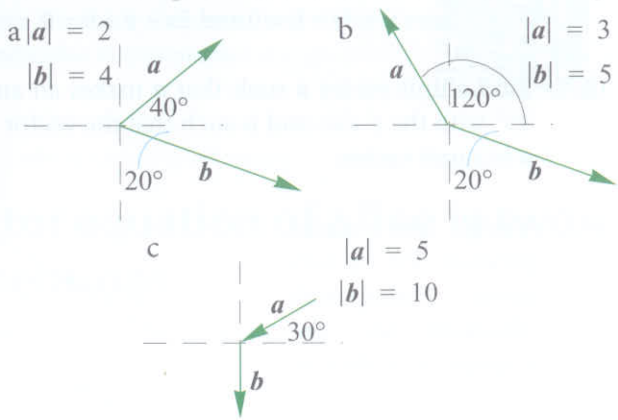



Scalar product calculations can be found by scrolling twice (using F6) to the right and pressing F2.



## Exercise C.11.5

1. Find the scalar product, $\boldsymbol{a} \bullet \boldsymbol{b}$, for each of the following:




2. Find the scalar products of these pairs of vectors.

a $3\boldsymbol{i} + 2\boldsymbol{j}$ and $2\boldsymbol{i} + 3\boldsymbol{j}$

b $3\boldsymbol{i} + 7\boldsymbol{j}$ and $2\boldsymbol{i} + 3\boldsymbol{j}$

c $3\boldsymbol{i} - \boldsymbol{j}$ and $-2\boldsymbol{i} + 2\boldsymbol{j}$

d $6\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$ and $-7\boldsymbol{i} - 4\boldsymbol{j} + 3\boldsymbol{k}$

e $-\boldsymbol{j} + 5\boldsymbol{k}$ and $-4\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$

f $-\boldsymbol{i} + 5\boldsymbol{j} + 4\boldsymbol{k}$ and $5\boldsymbol{i} - 4\boldsymbol{k}$

g $\begin{pmatrix} 0 \\ 6 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 7 \\ 2 \\ -6 \end{pmatrix}$

h $\begin{pmatrix} -3 \\ -1 \\ 7 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$

i $\begin{pmatrix} -6 \\ -1 \\ 7 \end{pmatrix}$ and $\begin{pmatrix} 7 \\ 3 \\ 5 \end{pmatrix}$

3. Find the angles between these pairs of vectors, giving the answers in degrees, correct to the nearest degree.

a $-4\boldsymbol{i} - 4\boldsymbol{j}$ and $-3\boldsymbol{i} + 2\boldsymbol{j}$

b $\boldsymbol{i} - \boldsymbol{j}$ and $3\boldsymbol{i} + 6\boldsymbol{j}$

c $-4\boldsymbol{i} - 2\boldsymbol{j}$ and $-\boldsymbol{i} - 7\boldsymbol{j}$

d $-7\boldsymbol{i} + 3\boldsymbol{j}$ and $-2\boldsymbol{i} - \boldsymbol{j}$

e $\boldsymbol{i} + 3\boldsymbol{j} + 7\boldsymbol{k}$ and $6\boldsymbol{i} + 7\boldsymbol{j} - \boldsymbol{k}$

f $\boldsymbol{j} + 3\boldsymbol{k}$ and $-\boldsymbol{j} - 2\boldsymbol{k}$

g $\begin{pmatrix} -3 \\ -1 \\ -5 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 5 \\ -5 \end{pmatrix}$

h $\begin{pmatrix} -2 \\ 7 \\ -7 \end{pmatrix}$ and $\begin{pmatrix} 5 \\ 2 \\ -5 \end{pmatrix}$

4. Two vectors are defined as $\boldsymbol{a} = 2\boldsymbol{i} + x\boldsymbol{j}$ and $\boldsymbol{b} = \boldsymbol{i} - 4\boldsymbol{j}$. Find the value of $x$ if:

a the vectors are parallel.

b the vectors are perpendicular.

5. If $\boldsymbol{a} = 2\boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{k}$, $\boldsymbol{b} = -\boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k}$ and $\boldsymbol{c} = \boldsymbol{i} + \boldsymbol{k}$, find, where possible,

a $\boldsymbol{a} \bullet \boldsymbol{b}$

b $(\boldsymbol{a} - \boldsymbol{b}) \bullet \boldsymbol{c}$

c $\boldsymbol{a} \bullet \boldsymbol{b} \bullet \boldsymbol{c}$

d $(\boldsymbol{a} - \boldsymbol{b}) \bullet (\boldsymbol{a} + \boldsymbol{b})$

e $\dfrac{\boldsymbol{a}}{\boldsymbol{c}}$

f $\boldsymbol{b} \bullet \boldsymbol{0}$

6. If $\boldsymbol{a} = 2\boldsymbol{i} - \sqrt{3}\boldsymbol{j}$, $\boldsymbol{b} = \sqrt{3}\boldsymbol{i} - \boldsymbol{j}$ and $\boldsymbol{c} = \boldsymbol{i} + \boldsymbol{j}$, find, where possible:

a $\boldsymbol{a} \bullet (\boldsymbol{b} + \boldsymbol{c}) + \boldsymbol{b} \bullet (\boldsymbol{c} - \boldsymbol{a}) + \boldsymbol{c} \bullet (\boldsymbol{a} - \boldsymbol{b})$

b $(\boldsymbol{b} - \boldsymbol{c}) \bullet (\boldsymbol{c} - \boldsymbol{b}) + |\boldsymbol{b}^2|$

c $2|\boldsymbol{a}|^2 - \sqrt{3}\boldsymbol{c} \bullet \boldsymbol{c}$

d $\sqrt{\dfrac{\boldsymbol{a}}{|\boldsymbol{a}|} + \dfrac{\boldsymbol{b}}{|\boldsymbol{b}|} + \dfrac{\boldsymbol{c}}{|\boldsymbol{c}|}}$

7. Find the value(s) of $x$ for which the vectors $x\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$ and $x\boldsymbol{i} - 2x\boldsymbol{j} - \boldsymbol{k}$ are perpendicular.

8. P, Q and R are three points in space with coordinates (2, −1, 4), (3, 1, 2) and (−1, 2, 5) respectively. Find angle Q in the triangle PQR.

9. Find the values of $x$ and $y$ if $\boldsymbol{u} = x\boldsymbol{i} + 2y\boldsymbol{j} - 8\boldsymbol{k}$ is perpendicular to both $\boldsymbol{v} = 2\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{w} = 3\boldsymbol{i} + 2\boldsymbol{j} - 4\boldsymbol{k}$.

10. Find the unit vector that is perpendicular to both $\boldsymbol{a} = 3\boldsymbol{i} + 6\boldsymbol{j} - \boldsymbol{k}$ and $\boldsymbol{b} = 4\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$.

11. Show that, if $\boldsymbol{u}$ is a vector in three dimensions, then $\boldsymbol{u} = (\boldsymbol{u} \bullet \boldsymbol{i})\boldsymbol{i} + (\boldsymbol{u} \bullet \boldsymbol{j})\boldsymbol{j} + (\boldsymbol{u} \bullet \boldsymbol{k})\boldsymbol{k}$.

12. a Find a vector perpendicular to both $\boldsymbol{a} = -\boldsymbol{i} + 2\boldsymbol{j} + 4\boldsymbol{k}$ and $\boldsymbol{b} = 2\boldsymbol{i} - 3\boldsymbol{j} + 2\boldsymbol{k}$.

b Find a vector perpendicular to $2\boldsymbol{i} + \boldsymbol{j} - 7\boldsymbol{k}$.

13. Show that if $|\boldsymbol{a} - \boldsymbol{b}| = |\boldsymbol{a} + \boldsymbol{b}|$, where $\boldsymbol{a} \neq \boldsymbol{0}$ and $\boldsymbol{b} \neq \boldsymbol{0}$, then $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular.

14. If $\boldsymbol{a} \bullet \boldsymbol{b} = \boldsymbol{a} \bullet \boldsymbol{c}$ where $\boldsymbol{a}$ ¦ $\boldsymbol{0}$ ¦ $\boldsymbol{b}$, what conclusion(s) can be made?

15. Using the scalar product for vectors prove that the cosine of the angle between two lines with direction cosines $l_1, m_1, n_1$ and $l_2, m_2, n_2$ is given by $\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2$.

16. Find the cosine of the acute angle between:

a two diagonals of a cube.

b the diagonal of a cube and one of its edges.

17. a On the same set of axes sketch the graphs of:

$x + 3y - 6 = 0$ and $2x - y + 6 = 0$,

clearly labelling all intercepts with the axes.

b Find a unit vector along the line:

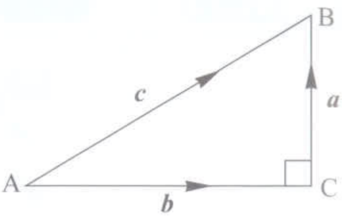
i $x + 3y - 6 = 0$.

ii $2x - y + 6 = 0$.

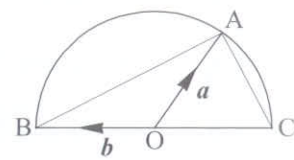
c Hence find the acute angle between the two lines $x + 3y - 6 = 0$ and $2x - y + 6 = 0$.

18. Find a unit vector $\boldsymbol{a}$ such that $\boldsymbol{a}$ makes an angle of 45° with the $z$-axis and is such that the vector $\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{a}$ is a unit vector.

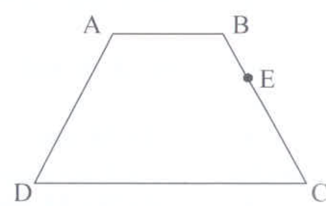
19. Using the scalar product for vectors prove Pythagoras's Theorem for the triangle ABC shown.




20 Prove that an angle inscribed in a semicircle is a right angle.




21. In the trapezium shown, BE:BC = 1:3.




Show that $3\mathbf{AC} \bullet \mathbf{DE} = 2(4m^2 - n^2)$

where $|\mathbf{AB}| = m$, $|\mathbf{DC}| = 2|\mathbf{AB}|$ and $|\mathbf{DA}| = n$

22. Prove that the altitudes of any triangle are concurrent.

23. An oil pipeline runs from a well (W) to a distribution point (D) which is 4 km east and 8 km north of the well. A second well (S) is drilled at a point 9 km east and 7 km south of the distribution point. It is desired to lay a new pipeline from the second well to a point (X) on the original pipeline where the two pipes will be joined. This new pipeline must be as short as possible.

a Set up a suitable vector basis using the first well as the origin.

b Express $\overrightarrow{WD}$, $\overrightarrow{WS}$, $\overrightarrow{DS}$ in terms of your basis.

c Write a unit vector in the direction of $\overrightarrow{WD}$.

d If the point X is $d$ km along the pipeline from the first well, write a vector equal to $\overrightarrow{WX}$.

e Hence find the vector $\overrightarrow{WX}$ such that the new pipeline is as short as possible.

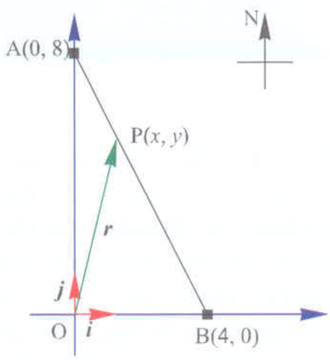
Link to a 3-d visualisation of two vectors, the plane in which they exist and a vector perpendicular to this plane.

# Vector equation of a line in two dimensions

We start this section by considering the following problem:

Relative to an origin O, a house, situated 8 km north of O, stands next to a straight road. The road runs past a second house, located 4 km east of O. If a person is walking along the road from the house north of O to the house east of O, determine the position of the person while on the road relative to O.

We start by drawing a diagram and place the person along the road at some point P. We need to determine the position vector of point P.




We have:

$\mathbf{r} = \mathbf{OP} = \mathbf{OA} = \mathbf{AP}$

Now, as P lies somewhere along $\mathbf{AB}$, we can write:

$\mathbf{AP} = \lambda \mathbf{AB}$, where $0 \leq \lambda \leq 1$, so that when $\lambda = 0$ the person is at A and when $\lambda = 1$ the person is at B.

Next, $\mathbf{AB} = \mathbf{AO} + \mathbf{OB} = -8\boldsymbol{j} + 4\boldsymbol{i}$, and so we have:

$\mathbf{r} = 8\boldsymbol{j} + \lambda(-8\boldsymbol{j} + 4\boldsymbol{i})$.

This provides us with the position vector of the person while walking on the road.

We take this equation a little further. The position vector of P can be written as $\mathbf{r} = x\boldsymbol{i} + y\boldsymbol{j}$ and so we have that

$x\boldsymbol{i} + y\boldsymbol{j} = 8\boldsymbol{j} + \lambda(-8\boldsymbol{j} + 4\boldsymbol{i})$

That is, we have $x\boldsymbol{i} + y\boldsymbol{j} = 4\lambda\boldsymbol{i} + (8 - 8\lambda)\boldsymbol{j}$ meaning that

$x = 4\lambda$ and $y = 8 - 8\lambda$.

The equations $x = 4\lambda$ – (1) and $y = 8 - 8\lambda$ – (2) are known as the **parametric form** of the equations of a straight line

Next, from these parametric equations, we have:

$\lambda = \dfrac{x}{4}$ – (3) and $\lambda = \dfrac{y-8}{-8}$ – (4)

Then, equating (3) and (4) we have $\frac{x}{4} = \frac{y-8}{-8}$. This equation is known as the Cartesian form of the equation of a straight line. We can go one step further and simplify this last equation.

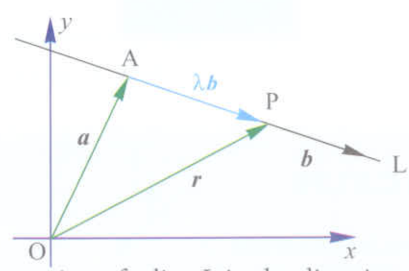
$$\frac{x}{4} = \frac{y-8}{-8} \Leftrightarrow -2x = y - 8 \Leftrightarrow y = -2x + 8$$

which corresponds to the Cartesian Equation of the straight line passing through A and B.

This approach to describe the position of an object (or person) is of great value when dealing with objects travelling in a straight line. When planes are coming in for landing, it is crucial that their positions along their flight paths are known, otherwise one plane could be heading for a collision with another plane in the air.

We now formalise the definition of the vector equation of a line in a plane:

The vector equation of a line L in the direction of the vector b, passing through the point A with position vector a is given by r = a + λb where λ is a scalar parameter.




The vector equation of a line L in the direction of the vector ***b***, passing through the point A with position vector **a** is given by:

$2\boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{k}$

where λ is a scalar parameter.

**Proof:**

Let the point P($x$, $y$) be any point on the line L, then the vector $\mathbf{AP}$ is parallel to the vector $\boldsymbol{b}$.

$$r = \mathbf{OP}$$
$$= \mathbf{OA} + \mathbf{AP}$$
$$\therefore r = a + \lambda b$$

So the equation of L is given by $r = a + \lambda b$ as required.

We can now derive two other forms for equations of a line. We start by letting the coordinates of A be $(a_1, a_2)$, the coordinates of P be $(x, y)$ and the vector $b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$.

From $r = a + \lambda b$ we have:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \lambda\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \Leftrightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_1 + \lambda b_1 \\ a_2 + \lambda b_2 \end{pmatrix}$$

This provides us with the:

**Parametric form** for the equation of a straight line:

$$x = a_1 + \lambda b_1 \quad y = a_2 + \lambda b_2$$

Next, from the parametric form we have:

$$x = a_1 + \lambda b_1 \Leftrightarrow x - a_1 = \lambda b_1 \Leftrightarrow \lambda = \frac{x - a_1}{b_1} \quad -(1)$$

and $$y = a_2 + \lambda b_2 \Leftrightarrow y - a_2 = \lambda b_2 \Leftrightarrow \lambda = \frac{y - a_2}{b_2} \quad -(2)$$

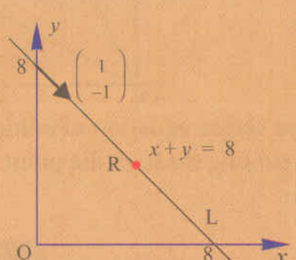
Equating (1) and (2) provides us with the:

**Cartesian form** for the equation of a straight line:

$$\frac{x - a_1}{b_1} = \frac{y - a_2}{b_2}$$

## Example C.11.16

Find the vector equation of the line L, as shown in the diagram. Comment on the uniqueness of this equation.




The vector equation of the line L is based on finding (or using) *any* point on the line, such as (0,8), and *any* vector in the direction of the line L, such as $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

The position vector of any point R on the line can then be written as $r = \begin{pmatrix} 0 \\ 8 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

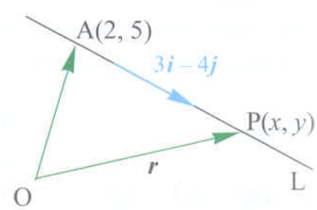
As $\lambda$ varies, different points on the line are generated, and conversely any point on the line has a corresponding value of $\lambda$. For example, substituting $\lambda = 3$ gives the point (3,5) and the point (8,0) corresponds to $\lambda = 8$.

NB: the vector equation (in parametric form) is not unique. The equation $r = \begin{pmatrix} 4 \\ 4 \end{pmatrix} + \lambda\begin{pmatrix} -2 \\ 2 \end{pmatrix}$ is an equally valid description of the line, and in this case substituting $\lambda = 0.5$ generates the point (3,5).

## Example C.11.17

Find the vector equation of the line L, passing through the point A(2, 5) and parallel to the vector $3i - 4j$.

Rather than depend on a standard formula, it is always helpful to visualise problems such as these. This is particularly useful when we move onto straight lines in space. We draw a general representation of this situation and work from there.




Let the point P be any point on the line L with position vector **r**, then $\mathbf{OP} = \mathbf{OA} + \mathbf{AP}$

However, as A and P lie on the line L, then $\mathbf{AP} = \lambda(3i - 4j)$.

Therefore, $r = (2i + 5j) + \lambda(3i - 4j)$

This represents the vector equation of the line L in terms of the parameter $\lambda$, where $\lambda \in \mathbb{R}$.

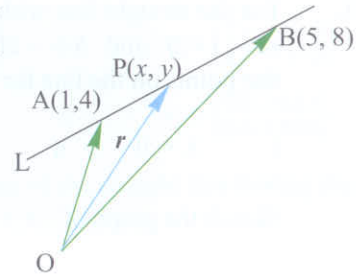
The equation could also be written as, $r = (2 + 3\lambda)i + (5 - 4\lambda)j$

## Example C.11.18

Find the vector equation of the line L, passing through the points A(1, 4) and B(5, 8). Give both the parametric form and Cartesian form of L.

We start with a sketch of the situation described:

Let the point P be any point on the line L with position vector $\boldsymbol{r}$, then




$$\mathbf{OP} = \mathbf{OA} + \mathbf{AP}$$

Then, as :

$$\mathbf{AP} \parallel \mathbf{AB} \Rightarrow \mathbf{AP} = l\,\mathbf{AB}$$

where $\lambda \in \mathbb{R}$.

This means that we need to find the vector $\mathbf{AB}$ which will be the vector parallel to the line L. So, we have:

$$\mathbf{AB} = \mathbf{AO} + \mathbf{OB} = -\begin{pmatrix}1\\4\end{pmatrix} + \begin{pmatrix}5\\8\end{pmatrix} = \begin{pmatrix}4\\4\end{pmatrix} = 4\begin{pmatrix}1\\1\end{pmatrix}$$

Therefore, from $\mathbf{OP} = \mathbf{OA} + \mathbf{AP}$ we have

$$\mathbf{OP} = \begin{pmatrix}1\\4\end{pmatrix} + \lambda \times 4\begin{pmatrix}1\\1\end{pmatrix}$$

That is, $\boldsymbol{r} = \begin{pmatrix}1\\4\end{pmatrix} + t\begin{pmatrix}1\\1\end{pmatrix}$ where $t = 4\lambda$

This represents the vector equation of the straight line L.

To find the **parametric form** of L we make use of the equation:

$$\boldsymbol{r} = \begin{pmatrix}1\\4\end{pmatrix} + t\begin{pmatrix}1\\1\end{pmatrix}.$$

As P(*x*, *y*) is any point on the line L, we write the vector equation as:

$$\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}1\\4\end{pmatrix} + t\begin{pmatrix}1\\1\end{pmatrix}.$$

From where we obtain the parametric equations, $x = 1 + t$ and $y = 4 + t$.

To find the **Cartesian form** of L we now make use of the parametric equations.

From $x = 1 + t$ we have $t = x - 1$ – (1) and from $y = 4 + t$ we have $t = y - 4$ – (2)

Then, equating (1) and (2) we have $x - 1 = y - 4$ (or $y = x + 3$ ).

## Example C.11.19

The vector equation of the line L, is given by $\boldsymbol{r} = \begin{pmatrix}3+2\lambda\\5-5\lambda\end{pmatrix}$

Express the vector equation in the standard form $\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{b}$.

Find a unit vector in the direction of L.

Find the Cartesian form of the line L.

$\boldsymbol{r} = \begin{pmatrix}3+2\lambda\\5-5\lambda\end{pmatrix} = \begin{pmatrix}3\\5\end{pmatrix} + \begin{pmatrix}2\lambda\\-5\lambda\end{pmatrix} = \begin{pmatrix}3\\5\end{pmatrix} + \lambda\begin{pmatrix}2\\-5\end{pmatrix}$ (which is in the form $\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{b}$ ).

The direction of the line L is provided by the vector $\boldsymbol{b}$, i.e. $\begin{pmatrix}2\\-5\end{pmatrix}$.

To find the unit vector we need $\left|\begin{pmatrix}2\\-5\end{pmatrix}\right| = \sqrt{4+25} = \sqrt{29}$.

$$\therefore \hat{\boldsymbol{b}} = \frac{1}{\sqrt{29}}\begin{pmatrix}2\\-5\end{pmatrix}.$$

Using the point P(*x*, *y*) as representing any point on the line L, we have that $\boldsymbol{r} = \begin{pmatrix}x\\y\end{pmatrix}$.

Therefore, we can write the vector equation as $\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}3+2\lambda\\5-5\lambda\end{pmatrix}$

From this equation we then have:

$$x = 3 + 2\lambda \text{ – (1) and } \quad y = 5 - 5\lambda \text{ – (2)}$$

We can now find the Cartesian equation by eliminating the parameter $\lambda$ using (1) and (2).

From (1): $\lambda = \frac{x-3}{2}$.

From (2): $\lambda = \frac{y-5}{-5}$

Therefore, $\frac{x-3}{2} = \frac{y-5}{-5}$.

## Example C.11.20

Find the angle between the lines $\frac{x-2}{4} = \frac{y+1}{3}$ and $\frac{x+2}{-1} = \frac{y-4}{2}$.

1. If finding the angles between two vectors, then the answer can either be acute or obtuse depending on the original arrangement between the two vectors.

2. If finding the angles between two lines, the answer should be stated as an acute angle, since we will 'create' two vectors from the lines and hence, depending on how we have created the vectors, the angle may be obtuse or acute.

We must first express the lines in their vector form. To do this we need to introduce a parameter for each line.

Let $\frac{x-2}{4} = \frac{y+1}{3} = \lambda$ giving the parametric equations:

$x = 2 + 4\lambda$ and $y = -1 + 3\lambda$ .

We can now express these two parametric equations in the vector form:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2+4\lambda \\ -1+3\lambda \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} + \lambda\begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

This vector equation informs us that the line $\frac{x-2}{4} = \frac{y+1}{3}$ is parallel to the vector $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$.

In the same way we can obtain the vector equation of the line:

$\frac{x+2}{-1} = \frac{y-4}{2}$ .

Let $\frac{x+2}{-1} = \frac{y-4}{2} = t$ giving the parametric equations:

$x = -2 - t$ and $y = 4 + 2t$ .

From here we obtain the vector equation:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2-t \\ 4+2t \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \end{pmatrix} + t\begin{pmatrix} -1 \\ 2 \end{pmatrix} .$$

This vector equation informs us that the line $\frac{x+2}{-1} = \frac{y-4}{2}$ is parallel to the vector $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$.

To find the angle between the two lines we use their direction vectors, $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ along with their scalar product:

$$\begin{pmatrix} 4 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \left|\begin{pmatrix} 4 \\ 3 \end{pmatrix}\right| \times \left|\begin{pmatrix} -1 \\ 2 \end{pmatrix}\right| \cos\theta$$

$$\Rightarrow -4 + 6 = \sqrt{16+9} \times \sqrt{1+4}\cos\theta$$

$$\Leftrightarrow \cos\theta = \frac{2}{5\sqrt{5}}$$

$$\therefore \theta \approx 79°42'$$

## Exercise C.11.6

1. For the straight line with equation $r = a + \lambda b$ where $a = i + 2j$ and $b = -2i + 3j$ , find the coordinates of the points on the line for which:

   i $\lambda = 0$ ii $\lambda = 3$ iii $\lambda = -2$

   Sketch the graph of $r = i + 2j + \lambda(-2i + 3j)$ .

2. Find the vector equation of the line passing through the point A and parallel to the vector $b$, where:

   a $A \equiv (2, 5)$ , $b = 3i - 4j$

   b $A \equiv (-3, 4)$ , $b = -i + 5j$

   c $A \equiv (0, 1)$ , $b = 7i + 8j$

   d $A \equiv (1, -6)$ , $b = 2i + 3j$

   e $A \equiv (-1, -1)$ , $b = \begin{pmatrix} -2 \\ 10 \end{pmatrix}$

   f $A \equiv (1, 2)$ , $b = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$

3. Find a vector equation of the line passing through the points A and B where:

   a $A(2, 3)$ , $B(4, 8)$

   b $A(1, 5)$ , $B(-2, 1)$

   c $A(4, -3)$ , $B(-1, -2)$

4. Find the vector equation of the straight line defined by the parametric equations:

   a $x = 9 + \lambda, y = 5 - 3\lambda$

   b $x = 6 - 4t, y = -6 - 2t$

   c $x = -1 - 4\lambda, y = 3 + 8\lambda$

   d $x = 1 + \frac{1}{2}\mu, y = 2 - \frac{1}{3}\mu$

5. Find the parametric form of the straight line having the vector equation:

a $r = \begin{pmatrix}-8\\10\end{pmatrix} + \mu\begin{pmatrix}2\\1\end{pmatrix}$ b $r = \begin{pmatrix}7\\4\end{pmatrix} + \mu\begin{pmatrix}-3\\-2\end{pmatrix}$

c $r = \begin{pmatrix}5\\3\end{pmatrix} + \frac{\mu}{2}\begin{pmatrix}5\\1\end{pmatrix}$ d $r = \begin{pmatrix}0.5-0.1t\\0.4+0.2t\end{pmatrix}$

6. Find the Cartesian form of the straight line having the vector equation:

a $r = \begin{pmatrix}1\\3\end{pmatrix} + \mu\begin{pmatrix}3\\1\end{pmatrix}$

b $r = \begin{pmatrix}2\\4\end{pmatrix} - \lambda\begin{pmatrix}7\\5\end{pmatrix}$

c $r = -\begin{pmatrix}2\\4\end{pmatrix} + \lambda\begin{pmatrix}1\\8\end{pmatrix}$

d $r = \begin{pmatrix}0.5\\0.2\end{pmatrix} - t\begin{pmatrix}-1\\11\end{pmatrix}$

e $r = \begin{pmatrix}7\\5\end{pmatrix} + \lambda\begin{pmatrix}0\\1\end{pmatrix}$

7. Write the following lines in vector form:

a $y = \frac{1}{3}x + 2$ b $y = x - 5$

c $2y - x = 6$

8. Find the position vector of the point of intersection of each pair of lines:

$r_1 = \begin{pmatrix}2\\1\end{pmatrix} + \begin{pmatrix}\lambda\\3\lambda\end{pmatrix}$ and $r_2 = \begin{pmatrix}1\\3\end{pmatrix} + \mu\begin{pmatrix}1\\2\end{pmatrix}$.

$r_1 = \begin{pmatrix}0\\4\end{pmatrix} + \lambda\begin{pmatrix}2\\5\end{pmatrix}$ and $r_2 = \begin{pmatrix}2\\-2\end{pmatrix} + \mu\begin{pmatrix}1\\1\end{pmatrix}$.

9. Find the equation of the line that passes through the point A (2, 7) and is perpendicular to the line with equation $r = -i - 3j + \lambda(3i - 4j)$ .

10. Let the position vectors of the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ be $p$ and $q$ respectively.

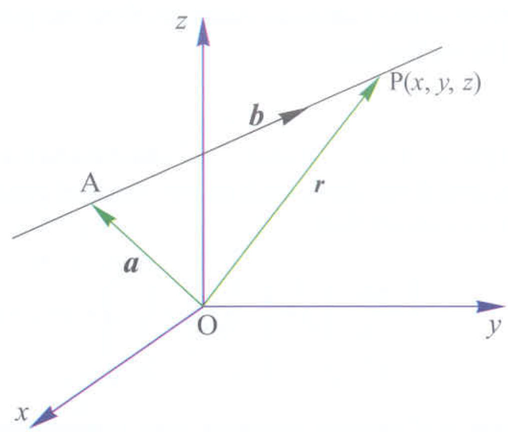
Show that the equation $r = (1-\lambda)p + \lambda q$ represents a vector equation of the line through P and Q, where $\lambda \in \mathbb{R}$.

Extra questions

# Lines in three dimensions

In three-dimensional work, always try to visualise situations very clearly. Because diagrams are never very satisfactory, it is useful to use the corner of a table with an imagined vertical line for axes; then pencils become lines and books or sheets of paper become planes.

It is tempting to generalise from a two-dimensional line like $x + y = 8$ and think that the Cartesian equation of a three dimensional line will have the form $x + y + z = 8$. This is not correct – as we will see later **this represents a plane, not a line**.




We approach lines in three dimensions in exactly the same way we did for lines in two dimensions. For any point P(x, y, z) on the line having the position vector $r$, passing through the point A and parallel to a vector in the direction of the line, $b$ say, we can write the equation of the line as $r = a + \lambda b$.

So, for example, the line passing through the point (4, 2, 5) and having the direction vector $i - j + 2k$ can be written as:

$$r = \begin{pmatrix}4\\2\\5\end{pmatrix} + \lambda\begin{pmatrix}1\\-1\\2\end{pmatrix}$$

Or, it could also have been written in $i, j, k$ form as

$$r = 4i + 2j + 5k + \lambda(i - j + 2k)$$

As for the case in 2–D, the parametric form or Cartesian form of the equation is obtained by using a point P(x, y, z) on the line with position vector:

$$r = \begin{pmatrix}x\\y\\z\end{pmatrix} \text{ so that } \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}4\\2\\5\end{pmatrix} + \lambda\begin{pmatrix}1\\-1\\2\end{pmatrix}$$

From here we first get the **parametric equations**:

$x = 4 + \lambda$, $y = 2 - \lambda$ and $z = 5 + 2\lambda$.

Solving each of these for $\lambda$, we get:

$$\lambda = x - 4 = 2 - y = \frac{z-5}{4}$$

The parameter $\lambda$ plays no part in the Cartesian equation, so we drop it and write the **Cartesian equation** as:

$$x - 4 = 2 - y = \frac{z-5}{4}.$$

It is important to be clear what this means: if we choose $x$, $y$ and $z$ satisfying the Cartesian equation, then the point $P(x, y, z)$ will be on the line.

For example $x = 10$, $y = -4$ and $z = 17$ satisfies the Cartesian equation, and if we think back to our original parametric equation we can see that:

$$\begin{pmatrix} 10 \\ -4 \\ 17 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 5 \end{pmatrix} + 6\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

To convert a Cartesian equation into parametric form we reverse the process and introduce a parameter $\lambda$. For example if the Cartesian equation is:

$\frac{x-1}{3} = \frac{y+2}{2} = \frac{z-6}{4}$ we write:

$$\frac{x-1}{3} = \frac{y+2}{2} = \frac{z-6}{4} = \lambda$$

$$x = 1 + 3\lambda$$

$$y = -2 + 2\lambda$$

$$z = 6 + 4\lambda$$

$$r = \begin{pmatrix} 1 \\ -2 \\ 6 \end{pmatrix} + \lambda\begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}$$

You will probably have noticed the strong connection between the numbers in the fractions in the Cartesian form and the numbers in the vectors in the parametric form.

Consider the **Cartesian form** of any straight line passing through the point $P(x_1, y_1, z_1)$ :

$$\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$$

From this equation we obtain the **parametric form** of the straight line:

$$\frac{x-x_1}{a} = \lambda \Rightarrow x = x_1 + \lambda a$$

$$\frac{y-y_1}{b} = \lambda \Rightarrow y = y_1 + \lambda b$$

$$\frac{z-z_1}{c} = \lambda \Rightarrow z = z_1 + \lambda c$$

which then leads to the **vector form** of the straight line:

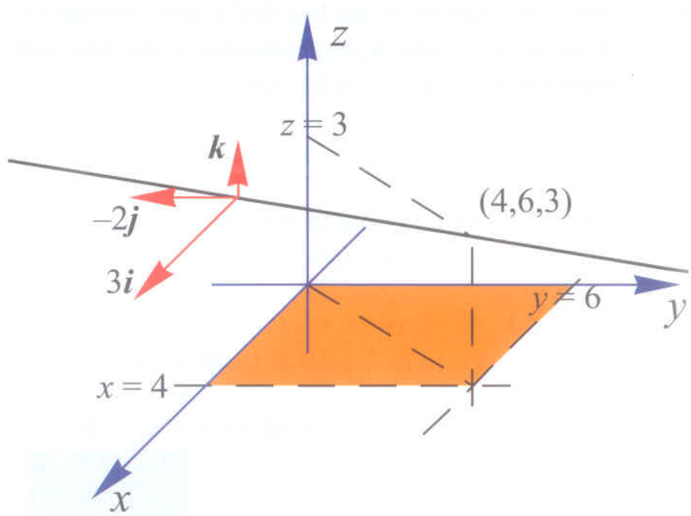
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \lambda\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

That is, the denominators of the Cartesian form of a straight line provide the coefficients of the directional vector of the line. This is an important observation, especially when finding the angle between two lines when the equation of the line is provided in Cartesian form. However, rather than simply committing this observation to memory, it is always a good idea to go through the (very short) working involved.

**Example C.11.21**

Find the Cartesian form of the straight line passing through the point (4, 6, 3) and having direction vector $3i - 2j + k$ . Draw a sketch of this line on a set of axes.

We start by sketching the line:




The direction vector of the line is $3i - 2j + k$ and as the line passes through the point (4, 6, 3), the vector equation of the line is given by $r = (4i + 6j + 3k) + \lambda(3i - 2j + k)$ .

From the vector equation we obtain the parametric form of the line: $x = 4 + 3\lambda, y = 6 - 2\lambda$ and $z = 3 + \lambda$ .

From these equations we have, $\lambda = \frac{x-4}{3}$ , $\lambda = \frac{y-6}{-2}$ and $\lambda = \frac{z-3}{1}$

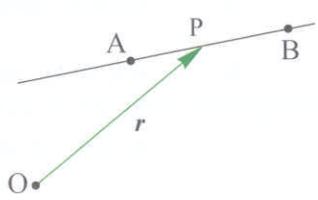
Then, eliminating $\lambda$ we have $\frac{x-4}{3} = \frac{y-6}{-2} = \frac{z-3}{1}$ or $\frac{x-4}{3} = \frac{y-6}{-2} = z - 3$

which represents the Cartesian form of the line.

### Example C.11.22

Find the vector form of the equation of the line through the point A(2, 1, 1) and the point B(4, 0, 3).

We make a very rough sketch – there is no point in trying to plot A and B accurately. Let the position vector of any point P on the line be $r$.




Then the vector form of the line is $r = \mathbf{OA} + \lambda\mathbf{AB}$.

Now, $\mathbf{OP} = r = \mathbf{OA} + \mathbf{AP}$ .

But $\mathbf{AP} = l\,\mathbf{AB} \therefore r = \mathbf{OA} + l\,\mathbf{AB}$ and

$\mathbf{AB} = \mathbf{AO} + \mathbf{OB} = -\mathbf{OA} + \mathbf{OB}$

$$\therefore \mathbf{AB} = -\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} \text{ and so,}$$

$$r = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$$

### Example C.11.23

Find the acute angle between the straight lines

$L_1: \frac{x-3}{2} = \frac{y+2}{-1} = \frac{z}{\sqrt{3}}$ and $L_1: \frac{x+1}{1} = \frac{y-2}{1} = \frac{z-1}{\sqrt{3}}$ .

Because the lines are given in their standard Cartesian form, we know that the denominators represent the coefficients of the direction vectors of these lines. As the angle between the lines is the same as the angle between their direction vectors we need only use the direction vectors of each line and then apply the dot product.

For $L_1$ the direction vector is $b_1 = 2i - j + \sqrt{3}k$ and for $L_2$ it is $b_2 = i + j + \sqrt{3}k$.

Using the dot product we have:

$b_1 \bullet b_2 = |b_1||b_2|\cos\theta$

$\therefore (2i - j + \sqrt{3}k) \bullet (i + j + \sqrt{3}k) = \sqrt{8} \times \sqrt{5}\cos\theta$

$2 - 1 + 3 = \sqrt{40}\cos\theta$

$\cos\theta = \frac{4}{\sqrt{40}}$

$\therefore \theta = 50°46'$

### Example C.11.24

Write the equation of the line $\frac{x+1}{3} = \frac{4-y}{2} = z$

in parametric form, and show that it is parallel to:

$-i + 5j + k + \mu(-6i + 4j - 2k)$.

From the Cartesian form of the line $\frac{x+1}{3} = \frac{4-y}{2} = z = \lambda$ we obtain the parametric form:

$x = -1 + 3\lambda, y = 4 - 2\lambda$ and $z = \lambda$.

We can then write this in the vector form

$r = -i + 4j + \lambda(3i - 2j + k)$.

Comparing the direction vectors of the two lines we see that:

$$-6i + 4j - 2k = -2(3i - 2j + k)$$

and so the direction vectors (and hence the lines) are parallel.

It is worth emphasising, that lines will be parallel or perpendicular if their direction vectors are parallel or perpendicular.

1. If the two lines are perpendicular we have $b_1 \bullet b_2 = 0 \Rightarrow x_1x_2 + y_1y_2 + z_1z_2 = 0$ .

2. If the two lines are parallel we have $b_1 = mb_2, m \neq 0$

**Example C.11.25**

Line L passes through the points (4, 3, 9) and (7, 8, 5), while line M passes through the points (12, 16, 4) and (*k*, 26,−4), where $k \in \mathbb{R}$. Find the value(s) of *k*, if:

a L is parallel to M.

b L is perpendicular to M.

We first need to determine direction vectors for both L and M.

For L: Let the points be A(4, 3, 9) and B(7, 8, 5), then a direction vector for L,

$b_1$ (for example), is given by $b_1 = \begin{pmatrix} 7-4 \\ 8-3 \\ 5-9 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ -4 \end{pmatrix}$ .

For M: Let the points be X(12, 16, 4) and Y(*k*, 26, −4), then a direction vector for M:

$b_2$ (for example), is given by $b_2 = \begin{pmatrix} k-12 \\ 26-16 \\ -4-4 \end{pmatrix} = \begin{pmatrix} k-12 \\ 10 \\ -8 \end{pmatrix}$ .

a If $L \parallel M$ we must have that $b_1 = cb_2, c \in \mathbb{R}$.

i.e. $\begin{pmatrix} 3 \\ 5 \\ -4 \end{pmatrix} = c\begin{pmatrix} k-12 \\ 10 \\ -8 \end{pmatrix} \Rightarrow \frac{3}{k-12} = \frac{5}{10} = -\frac{4}{-8}$

So that $\frac{3}{k-12} = \frac{1}{2} \Leftrightarrow k - 12 = 6 \Leftrightarrow k = 18$ .

b If L is perpendicular to M, we must have that $b_1 \bullet b_2 = 0$ .

i.e. $\begin{pmatrix} 3 \\ 5 \\ -4 \end{pmatrix} \bullet \begin{pmatrix} k-12 \\ 10 \\ -8 \end{pmatrix} = 0 \Rightarrow 3(k-12) + 50 + 32 = 0$

$\Leftrightarrow 3k = -46 \Leftrightarrow k = -\frac{46}{3}$

## Exercise C.11.7

1. Find the vector form of the line passing through the point:

   a A(2, 1, 3) which is also parallel to the vector $i - 2j + 3k$ .

   b A(2, −3, −1) which is also parallel to the vector $-2i + k$ .

2. Find the vector form of the line passing through the points:

   a $A(2, 0, 5)$ and $B(3, 4, 8)$ .

   b $A(3, -4, 7)$ and $B(7, 5, 2)$ .

   c $A(-3, 4, -3)$ and $B(4, 4, 4)$.

3. Find the Cartesian form of the line having the vector form:

   a $r = \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} + s\begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}$

   b $r = \begin{pmatrix} -2 \\ 3 \\ -1 \end{pmatrix} + t\begin{pmatrix} 5 \\ 0 \\ -2 \end{pmatrix}$ c $r = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + s\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

4. Find the Cartesian equation of the line passing through the points A(5, 2, 6) and B(−2, 4, 2). Also, provide the parametric form of this line.

5. For the line defined by the parametric equations $x = 3 + 2t, y = 4 - 3t$ and $z = 1 + 5t$ , find the coordinates of where the line crosses the *xy*-plane.

6. Convert these lines to their parametric form:

a $\frac{x-2}{3}=y-5=2(z-4)$

b $\frac{2x-1}{3}=y=\frac{4-z}{2}$

c $\frac{x-3}{-1}=\frac{2-y}{3}=\frac{z-4}{2}$

d $\frac{2x-2}{4}=\frac{3-y}{-2}=\frac{2z-4}{1}$

7. Convert these lines to their Cartesian form:

a $\boldsymbol{r}=\begin{pmatrix}4\\1\\-2\end{pmatrix}+t\begin{pmatrix}3\\-4\\-2\end{pmatrix}$

b $\boldsymbol{r}=2\boldsymbol{i}+\boldsymbol{k}+\mu(\boldsymbol{j}-3\boldsymbol{k})$

8. Show that the lines $\frac{x-1}{2}=2-y=5-z$ and $\frac{4-x}{4}=\frac{3+y}{2}=\frac{5+z}{2}$ are parallel.

9. Find the Cartesian equation of the lines joining the points

a (–1, 3, 5) to (1, 4, 4)

b (2, 1, 1) to (4, 1, –1)

10. a Find the coordinates of the point where the line: $\boldsymbol{r}=\begin{pmatrix}-2\\5\\3\end{pmatrix}+t\begin{pmatrix}-1\\2\\1\end{pmatrix}$ intersects the $x$-$y$ plane.

b The line $\frac{x-3}{4}=y+2=\frac{4-z}{5}$ passes through the point $(a, 1, b)$. Find the values of $a$ and $b$.

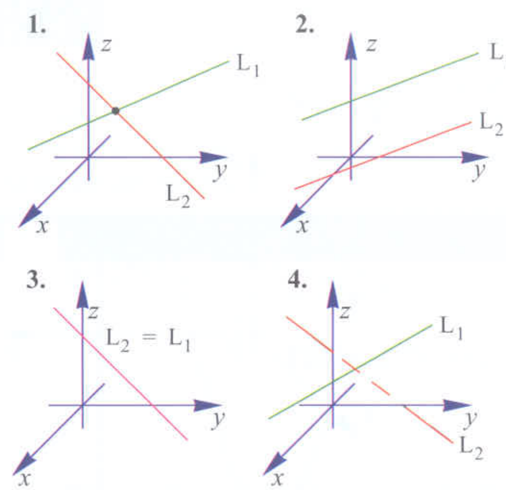
Extra questions

## Intersection of two lines in 3–D

Two lines in space may:

1. intersect at a point, or
2. be parallel and never intersect, or
3. be parallel and coincident (i.e. the same), or
4. be neither parallel nor intersect.

Of the above scenarios, the first three are consistent with our findings when dealing with lines in a plane (i.e. 2-D), however, the fourth scenario is new. We illustrate these now.




Two lines that meet at (at least) one point must lie in the same plane (cases **1** and **3**). Two intersecting lines or two parallel lines are said to be coplanar (cases **1**, **2** and **3**). Two lines which are not parallel and which do not intersect are said to be **skew**. Skew lines do not lie on the same plane, i.e. they are not coplanar (case **4**).

Lines lying on the $xy$–, $xz$– and $yz$– planes

From the Cartesian form of the straight line,

L: $\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}$ we can write:

$$\frac{x-x_1}{a}=\frac{y-y_1}{b}\Leftrightarrow b(x-x_1)=a(y-y_1) \text{ – (1)}$$

$$\frac{x-x_1}{a}=\frac{z-z_1}{c}\Leftrightarrow c(x-x_1)=a(z-z_1) \text{ – (2)}$$

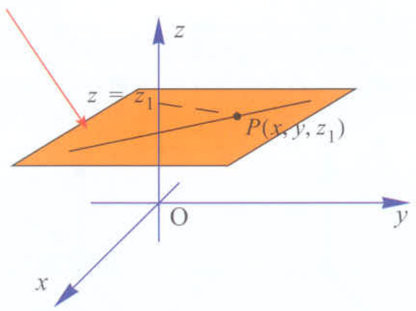
$$\frac{y-y_1}{b}=\frac{z-z_1}{c}\Leftrightarrow c(y-y_1)=b(z-z_1) \text{ – (3)}$$

Equations (1), (2) and (3) represent the planes perpendicular to the $xy$-, $xz$- and $yz$ planes respectively. Each of these equations is an equation of a plane containing L. The simultaneous solution of any pair of these planes will produce the same line. In fact, the three equations are not independent because any one of them can be derived from the other two. If any one of the numbers $a$, $b$ or $c$ is zero we obtain a line lying in one of the $xy$-, $xz$- or $yz$ planes. For example, consider the case that $c = 0$ and neither $a$ nor $b$ is zero.

In such a case we have, $\frac{x-x_1}{a} = \frac{y-y_1}{b}$ and $z = z_1$ meaning that the line lies on the plane containing the point $z = z_1$ and parallel to the $xy$-plane.

3-d image showing that skew lines may appear to intersect from some viewpoints.

$\frac{x-x_1}{a} = \frac{y-y_1}{b}$ and $z = z_1$




For 'convenience' we sometimes write the equation as

$\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{0}$, although

clearly, $\frac{z-z_1}{0}$ has no meaning.

**Example C.11.26**

Line L passes through the points A(1, 2, –1) and B(11, –2, –7) while line M passes though the points C(2, –1, –3) and D(9, –10, 3). Show that L and M are skew lines.

We start by finding the vector equations of both lines. For L we have a direction vector given by $\boldsymbol{b}_1 = (11-1)\boldsymbol{i} + (-2-2)\boldsymbol{j} + (-7-(-1))\boldsymbol{k} = 10\boldsymbol{i} - 4\boldsymbol{j} - 6\boldsymbol{k}$.

Then, as L passes through A(1, 2, –1), it has a vector equation given by $\boldsymbol{r} = \boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k} + \lambda(10\boldsymbol{i} - 4\boldsymbol{j} - 6\boldsymbol{k})$

This gives the parametric form as, $x = 1 + 10\lambda, y = 2 - 4\lambda$ and $z = -1 - 6\lambda$ – (1)

Similarly, we can find the parametric form for M.

The vector form of M is given by:

$\boldsymbol{r} = 2\boldsymbol{i} - \boldsymbol{j} - 3\boldsymbol{k} + \mu(7\boldsymbol{i} - 9\boldsymbol{j} + 6\boldsymbol{k})$

so the parametric form is given by $x = 2 + 7\mu, y = -1 - 9\mu$ and $z = -3 + 6\mu$ – (2)

Now, as the set of coefficients of the direction vector of M and L are not proportional, i.e. as $\frac{10}{7} \neq \frac{-4}{-9} \neq \frac{-6}{6}$, the lines L and M are not parallel.

Then, for the lines to intersect, there must be a value of λ and μ that will provide the same point $(x_0,y_0,z_0)$ lying on both L and M. Using (1) and (2) we equate the coordinates and try to determine this point $(x_0,y_0,z_0)$:

$1 + 10\lambda = 2 + 7\mu$ – (3)

$2 - 4\lambda = -1 - 9\mu$ – (4)

$-1 - 6\lambda = -3 + 6\mu$ – (5)

Solving for λ and μ using (4) and (5) we obtain:

$\mu = -\frac{5}{39}$ and $\lambda = \frac{18}{39}$.

Substituting these values into (1), we have

L.H.S = $1 + 10 \times \frac{18}{39} \neq 2 + 7 \times -\frac{5}{39}$ = R.H.S.

As the first equation is not consistent with the other two, the lines do not intersect and, as they are not parallel, they must be skew.

The techniques we have been discussing can be used to solve problems in particle motion (kinematics).

**Example C.11.27**

Particle A moves at a constant velocity from the point (1,2,3) to the point (21,32,23) over a period of 10 seconds. Particle B moves from (5,18,7) to (15,8,17) over the same period. Will the particles collide?

Particle A is translated (over 10 seconds): $\begin{pmatrix} 21-1 \\ 32-2 \\ 23-3 \end{pmatrix} = \begin{pmatrix} 20 \\ 30 \\ 20 \end{pmatrix}$

This represents a velocity vector of $\begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}$ (per sec).

The position of particle A is: $\boldsymbol{r}_A = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + t\begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}$

Particle B is translated (over 10 seconds):

$$\begin{pmatrix} 15-5 \\ 8-18 \\ 17-7 \end{pmatrix} = \begin{pmatrix} 10 \\ -10 \\ 10 \end{pmatrix}$$

This represents a velocity vector of $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ (per sec).

The position of particle B is: $\boldsymbol{r}_B = \begin{pmatrix} 5 \\ 18 \\ 7 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

If the particles collide, there is a time at which they are in the same position. This means that there is a value of $t$ such that:

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + t\begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 18 \\ 7 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

This means that:
$$1+2t=5+t \Rightarrow t=4$$
$$2+3t=18-t \Rightarrow t=4$$
$$3+2t=7+t \Rightarrow t=4$$

and the particles collide after 4 seconds.

### Example C.11.28

Two aircraft are flying in the vicinity of an aerodrome. The positions of the two aircraft relative to the ground radar are given by:

$$\boldsymbol{r}_A = \begin{pmatrix} 1 \\ 12 \\ 5 \end{pmatrix} + t\begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix} \quad \boldsymbol{r}_B = \begin{pmatrix} 4 \\ 5 \\ 3 \end{pmatrix} + t\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

Distance units are nautical miles and time is in minutes.

Find the vector that represents the separation of the two aircraft. Find the distance of closest approach and when this occurs.

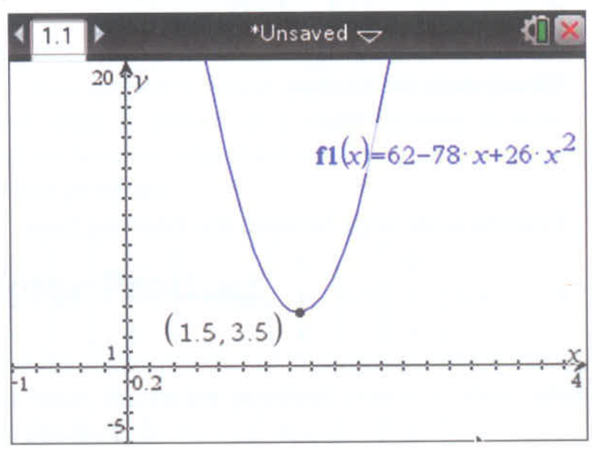
Vector from A to B is:

$$\boldsymbol{r}_B - \boldsymbol{r}_A = \begin{pmatrix} 4 \\ 5 \\ 3 \end{pmatrix} + t\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 12 \\ 5 \end{pmatrix} - t\begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix}$$
$$= \begin{pmatrix} 3 \\ -7 \\ -2 \end{pmatrix} + t\begin{pmatrix} -3 \\ 4 \\ 1 \end{pmatrix}$$

The distance between the aircraft is the absolute value of this function. We will work with the square of this absolute value:

$$|\boldsymbol{r}_B - \boldsymbol{r}_A|^2 = (3-3t)^2 + (-7+4t)^2 + (-2+t)^2$$
$$= 9 - 18t + 9t^2 + 49 - 56t + 16t^2 + 4 - 4t + t^2$$
$$= 62 - 78t + 26t^2$$

We can look for the time at which this expression is a minimum. This is because the square root function is one to one and increasing. We are after the minimum and can use a graph to find it. As with many 'applications' questions, it is necessary to adjust the graph window. We have used Analyze Graph to locate the minimum.




The closest approach occurs at $t$ = 1.5 and is $\sqrt{3.5}$ or about 1.9nm. Note also that one of the pilots will need to pay attention to avoid hitting the ground!

### Exercise C.11.8

1. Find the Cartesian equation of the lines joining the points

   a (−1, 3, 5) to (1, 4, 4)

   b (2, 1, 1) to (4, 1, −1)

2. Find the coordinates of the point where the line:

a $r = \begin{pmatrix} -2 \\ 5 \\ 3 \end{pmatrix} + t\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$ intersects the $x$-$y$ plane.

b The line $\frac{x-3}{4} = y+2 = \frac{4-z}{5}$ passes through the point $(a, 1, b)$. Find the values of $a$ and $b$.

3. Find the Cartesian equation of the line having the vector form:

a $r = \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ b $r = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + t\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$.

In each case, provide a diagram showing the lines.

4. Find the vector equation of the line represented by the Cartesian form $\frac{x-1}{2} = \frac{1-2y}{3} = z-2$ .

Clearly describe this line.

5. Find the acute angle between the following lines.

a $r = \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} + s\begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}$ and $s = \begin{pmatrix} -2 \\ 5 \\ 3 \end{pmatrix} + t\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$

b $r = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} + s\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$ and $s = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$

c $\frac{x-3}{-1} = \frac{2-y}{3} = \frac{z-4}{2}$ and $\frac{2x-1}{3} = \frac{y+5}{3} = \frac{z-1}{-2}$

6. Find the point of intersection of the lines:

a $\frac{x-5}{-2} = y-10 = \frac{z-9}{12}$ - $x = 4, \frac{y-9}{-2} = \frac{z+9}{6}$

b $\frac{2x-1}{3} = \frac{y+5}{3} = \frac{z-1}{-2}$ - $\frac{2-x}{4} = \frac{y+3}{2} = \frac{4-2z}{1}$

7. Find the Cartesian form of the lines with parametric equation given by:

L : $x = \lambda, y = 2\lambda + 2, z = 5\lambda$ and

M : $x = 2\mu - 1, y = -1 + 3\mu, z = 1 - 2\mu$

a Find the point of intersection of these two lines.

b Find the acute angle between these two lines.

Find the coordinates of the point where:

i L cuts the $x$-$y$ plane.

ii M cuts the $x$-$y$ plane.

8. Show that the lines $\frac{x-2}{3} = \frac{y-3}{-2} = \frac{z+1}{5}$ and $\frac{x-5}{-3} = \frac{y-1}{2} = \frac{z-4}{-5}$ are coincident.

9. Show that the lines $\frac{x-1}{-3} = y-2 = \frac{7-z}{11}$ and $\frac{x-2}{3} = \frac{y+1}{8} = \frac{z-4}{-7}$ are skew.

10. Find the equation of the line passing through the origin and the point of intersection of the lines with equations

$x - 2 = \frac{y-1}{4}, z = 3$ and $\frac{x-6}{2} = y - 10 = z - 4$ .

11. The lines $\frac{x}{3} = \frac{y-2}{4} = 3 + z$ and $x = y = \frac{z-1}{2k}$ , $k \in \mathbb{R} \backslash \{0\}$ meet at right angles. Find $k$.

12. Consider the lines L : $x = 0, \frac{y-3}{2} = z + 1$ and M : $\frac{x}{4} = \frac{y}{3} = \frac{z-10}{-1}$ .

Find, correct to the nearest degree, the angle between the lines L and M.

13. Find the value(s) of $k$, such that the lines:

$\frac{x-2}{k} = \frac{y}{2} = \frac{3-z}{3}$ and $\frac{x}{k-1} = \frac{y+2}{3} = \frac{z}{4}$ are

perpendicular.

14. Find a direction vector of the line that is perpendicular to both:

$$\frac{x+1}{3} = \frac{y+1}{8} = \frac{z+1}{12} \quad \text{and} \quad \frac{1-2x}{-4} = \frac{3y+1}{9} = \frac{z}{6}.$$

15. Are the lines:

$$\frac{x-1}{5} = \frac{y+2}{4} = \frac{4-z}{3} \quad \text{and} \quad \frac{x+2}{3} = \frac{y+7}{2} = \frac{2-z}{3}$$

parallel? Find the point of intersection of these lines. What do you conclude?

16. Two particles have position vectors:

$$r_A = \begin{pmatrix} 2 \\ 2 \\ 10 \end{pmatrix} + t\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$$

$$r_B = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}$$

Find when the particles collide.

17. Find the point of coincidence of:

$$r_A = \begin{pmatrix} -11 \\ 17 \\ -7 \end{pmatrix} + t\begin{pmatrix} 3 \\ -4 \\ 2 \end{pmatrix}$$

$$r_B = \begin{pmatrix} -1 \\ -5 \\ -3 \end{pmatrix} + t\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$$

Will the particles collide?

18. Find the closest approach of these two particles:

$$r_A = \begin{pmatrix} -4 \\ -2 \\ -3 \end{pmatrix} + t\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

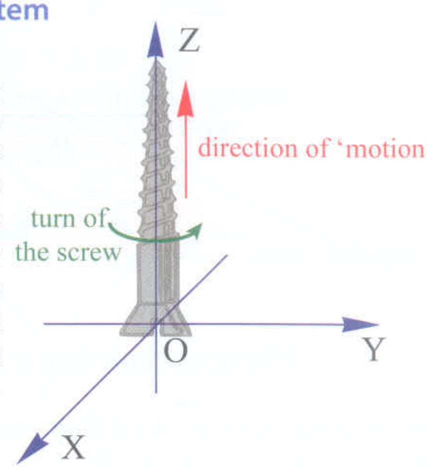
$$r_B = \begin{pmatrix} 9 \\ 5 \\ 8 \end{pmatrix} + t\begin{pmatrix} -1 \\ -2 \\ -2 \end{pmatrix}$$

# Three-dimensional Geometry

We start this section by establishing a definition:

## Right-handed system

When dealing with three-dimensional space, three base vectors (not coplanar) must be defined. We also conveniently use base vectors that are mutually orthogonal (at right-angles) and which are **right-handed**.




So, what do we mean by right-handed?

If we place a screw at some origin O and rotate it from OX to OY, then the screw would move in the direction OZ. This defines what is known as a **right-handed system**. This definition becomes important when we look at the operation of **vector product**.

# Vector Product

Unlike the scalar product of two vectors, which results in a scalar value, the **vector product** or as it is often called, the **cross product**, produces a vector.

We define the vector product as follows:

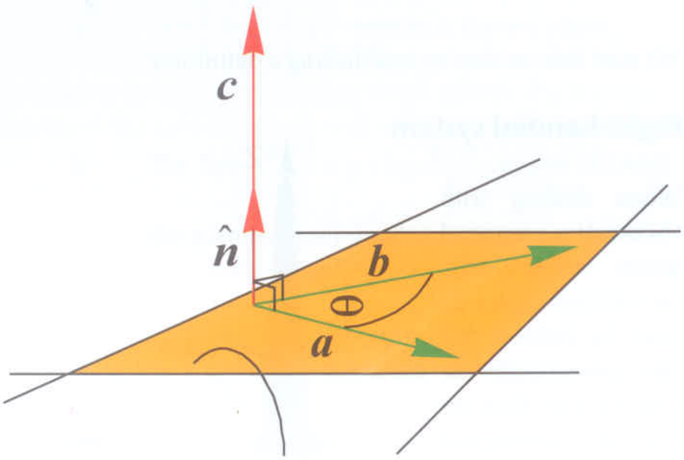
The vector product (or cross product) of two vectors, $\boldsymbol{a}$ and $\boldsymbol{b}$ produces a third vector, $\boldsymbol{c}$, where

$$\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\sin\theta\,\hat{\boldsymbol{n}}$$

and $\theta$ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ and $\hat{\boldsymbol{n}}$ is a unit vector perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$, i.e. to the plane of $\boldsymbol{a} \times \boldsymbol{b}$. This means that the vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\hat{\boldsymbol{n}}$ (**in that order**) form a right-handed system.

We now consider some properties of the vector product.

### Direction of $a \times b$




Plane containing $a$ and $b$

The resulting vector, $c = a \times b$ is a vector that is parallel to the unit vector $\hat{n}$ (unless $a \times b = \mathbf{0}$).

The direction of $\hat{n}$ (and hence $c$) is always either:

1. perpendicular to the plane containing $a$ and $b$ which is **determined by the right-hand rule** (as shown in the diagram).

or

2. is the zero vector, $\mathbf{0}$.

### Magnitude of $a \times b$

The magnitude of $a \times b$ is given by $|a \times b| = ||a||b|\sin\theta\,\hat{n}|$

$$= |a||b||\sin\theta||\hat{n}|$$

But, $|\hat{n}| = 1$ and $0 \le \theta \le \pi \Rightarrow \sin\theta \ge 0$, therefore, we have that:

$$|a \times b| = |a||b|\sin\theta$$

Notice that from 1 and 2, we can also conclude that:

If $a \times b = \mathbf{0}$, then either:

1. $a = \mathbf{0}$ or $b = \mathbf{0}$ or both $a$ and $b$ are $\mathbf{0}$ or

2. $\sin\theta = 0 \Rightarrow \theta = 0$ or $\pi$ (as $0 \le \theta \le \pi$).

Observation 2, i.e. $\sin\theta = 0 \Rightarrow \theta = 0$ or $\pi$, implies that $a$ and $b$ would be either parallel or antiparallel, which would not define a plane and so, the unit vector would not be defined.

This means that for any vector, $a$, $a \times a = \mathbf{0}$, which brings up a very interesting result for our $i$-$j$-$k$ – vector system:

$$i \times i = j \times j = k \times k = \mathbf{0}$$

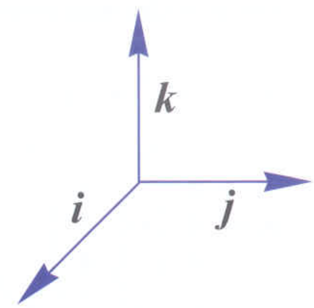
So, unlike the scalar product, where $a \bullet a = |a|^2 > 0$ for a non-zero vector $a$, with the cross product we have $a \times a = \mathbf{0}$. Also, recall that with the dot product, if the vectors $a$ and $b$ are non-zero and perpendicular, then $a \bullet b = 0$. So, what can we conclude about the cross product of two non-zero perpendicular vectors?

If the non-zero vectors $a$ and $b$ are perpendicular then

$\theta = \frac{\pi}{2} \Rightarrow \sin\theta = 1 \therefore a \times b = |a||b|\hat{n}$.

This means that the magnitude of $|a \times b| = |a||b||\hat{n}| = |a||b|$.

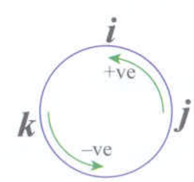
As a result of this property, we have for our $i$-$j$-$k$ – vector system the following results:




$i \times j = k, j \times k = i, k \times i = j$
and
$i \times k = -j, j \times i = -k, k \times j = -i$

The reason for the negative signs in the above is to ensure consistency within the right-hand system.

So that for example, the vectors $i$, $j$ and $k$ (in that order) form a right-hand system as do the vectors $i$, $k$ and $-j$ (in that order). A useful way of remembering which sign applies is to use the cyclic diagram shown:




1. Going **clockwise**, we take the **positive sign**, e.g. $k \times i = j$
2. Going **anticlockwise**, we take the **negative sign**, e.g. $j \times i = -k$

## Operational properties

### Closure

As $a \times b$ produces a unique vector, then the operation of vector product is closed.

### Commutativity

As $\boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{b} \times \boldsymbol{a}$ (to conform with the right-hand system) the operation of vector product is **not** commutative.

In fact, because of the change in sign, we say that the vector product is anti-commutative.

Notice also that $|\boldsymbol{a} \times \boldsymbol{b}| = |-\boldsymbol{b} \times \boldsymbol{a}| = |\boldsymbol{b} \times \boldsymbol{a}|$, i.e. the vector $\boldsymbol{a} \times \boldsymbol{b}$ has the same magnitude as $\boldsymbol{b} \times \boldsymbol{a}$ but is in the opposite direction.

### Associativity

You should try to verify that $(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c} \neq \boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c})$ (e.g. use $\boldsymbol{a} = \boldsymbol{i}$, $\boldsymbol{b} = \boldsymbol{j}$ and $\boldsymbol{c} = \boldsymbol{k}$) and so the vector product is **non-associative**.

### Distributivity

Also, try to verify that $\boldsymbol{a} \times (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \times \boldsymbol{b} + \boldsymbol{a} \times \boldsymbol{c}$ and as such, the vector product **is distributive over addition**.

### Identity

**No identity** element exists for the operation of vector product.

### Inverse

**No inverse** element exists for the operation of vector product.

## Exercise C.11.9

1. For each pair of coplanar vectors, find the magnitude of their cross product.

   a $|\boldsymbol{a}| = 5$, $|\boldsymbol{b}| = 2$ and the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ is 30°.

   b $|\boldsymbol{u}| = 1$, $|\boldsymbol{v}| = 8$ and the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$ is 60°.

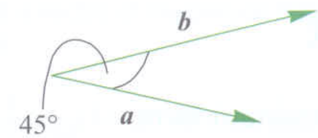
   c $|\boldsymbol{a}| = 3$ and $|\boldsymbol{b}| = 4$ where $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel.

   d $|\boldsymbol{u}| = 0.5$, $|\boldsymbol{v}| = 12$, where $\boldsymbol{u}$ and $\boldsymbol{v}$ are perpendicular.

   e $|\boldsymbol{a}| = 7$, $|\boldsymbol{b}| = 3$ and $\boldsymbol{a}$ and $\boldsymbol{b}$ are anti-parallel.

2. Sketch the following cross products for each pair of coplanar vectors:

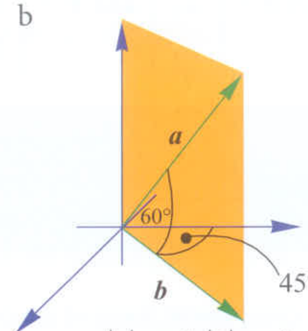
   a




$|\boldsymbol{a}| = 2$, $|\boldsymbol{b}| = 3$

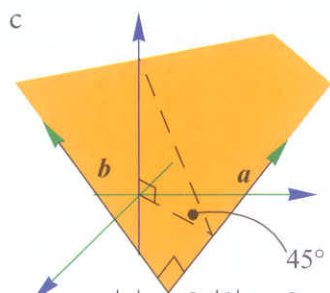
Where $\boldsymbol{a}$ and $\boldsymbol{b}$ can be considered as lying on the surface of an upright table.

   b




$|\boldsymbol{a}| = 4$, $|\boldsymbol{b}| = 3$

   c




$|\boldsymbol{a}| = 2$, $|\boldsymbol{b}| = 2$

   i $\boldsymbol{a} \times \boldsymbol{b}$ ii $\boldsymbol{b} \times \boldsymbol{a}$ iii $\boldsymbol{a} \times \boldsymbol{a}$

3. a If $|\boldsymbol{a}| = 5$, $|\boldsymbol{b}| = 4$ and $\boldsymbol{a} \bullet \boldsymbol{b} = 6$, find $|\boldsymbol{a} \times \boldsymbol{b}|$.

   b If $|\boldsymbol{a}| = 5$, $|\boldsymbol{b}| = 4$ and $\boldsymbol{a} \bullet \boldsymbol{b} = 12$, find $|\boldsymbol{a} \times \boldsymbol{b}|$

4. If $|\boldsymbol{a}| = 2$, $|\boldsymbol{b}| = 9$ and $|\boldsymbol{a} \times \boldsymbol{b}| = 15$, find the angle between the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.

5. If $|\boldsymbol{a}| = 3$, $|\boldsymbol{b}| = 3$ and $|\boldsymbol{a} \times \boldsymbol{b}| = 6$, find $\boldsymbol{a} \bullet \boldsymbol{b}$.

6. If $|\boldsymbol{a}| = 1$, $|\boldsymbol{b}| = \sqrt{3}$ where $\boldsymbol{a}$ and $\boldsymbol{b}$ are mutually perpendicular, find:

   a $|(\boldsymbol{a} + \boldsymbol{b}) \times (\boldsymbol{a} - \boldsymbol{b})|$.

   b $|(2\boldsymbol{a} + \boldsymbol{b}) \times (\boldsymbol{a} - 2\boldsymbol{b})|$.

# Vector form of the Vector Product

## 1. Component form

The vector product is only defined when both vectors are three dimensional.

The vector product of $\boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ is given by:

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}$$

This is known as the **component form** of the cross product. The result is a third vector that is at right angles to the two original vectors. This can be verified by making use of the dot product. Using the 'product' $\boldsymbol{a} \bullet (\boldsymbol{a} \times \boldsymbol{b})$ we have:

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \bullet \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}$$

$$= a_1(a_2b_3 - a_3b_2) + a_2(a_3b_1 - a_1b_3) + a_3(a_1b_2 - a_2b_1)$$
$$= a_1a_2b_3 - a_1a_3b_2 + a_2a_3b_1 - a_2a_1b_3 + a_3a_1b_2 - a_3a_2b_1$$
$$= 0$$

You should check for yourself that the vector product is also perpendicular to the second vector.

Also, notice that in the above diagram, the resulting vector $\boldsymbol{c}$, points in the direction that is consistent with the right-hand rule.

**Example C.11.29**

Find the vector product $\begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix} \times \begin{pmatrix} -1 \\ 4 \\ -2 \end{pmatrix}$.

$$\begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix} \times \begin{pmatrix} -1 \\ 4 \\ -2 \end{pmatrix} = \begin{pmatrix} 4 \times -2 - 1 \times 4 \\ 1 \times -1 - (-2) \times 2 \\ 2 \times 4 - (-1) \times 4 \end{pmatrix} = \begin{pmatrix} -12 \\ 3 \\ 12 \end{pmatrix}$$

Check:

$$\begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} -12 \\ 3 \\ 12 \end{pmatrix} = -24 + 12 + 12 = 0,$$

$$\begin{pmatrix} -1 \\ 4 \\ -2 \end{pmatrix} \bullet \begin{pmatrix} -12 \\ 3 \\ 12 \end{pmatrix} = 12 + 12 - 24 = 0$$

## 2. The Determinant form

When vectors are given in base vector notation, a more convenient method of finding the **Vector Cross Product** relies on a determinant representation. Given two vectors $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$ and $\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}$, the vector product $\boldsymbol{a} \times \boldsymbol{b}$ is defined as:

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\boldsymbol{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\boldsymbol{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\boldsymbol{k}$$

Applying this to the vectors in Example C.11.29, where $\boldsymbol{a} = 2\boldsymbol{i} + 4\boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{b} = -\boldsymbol{i} + 4\boldsymbol{j} - 2\boldsymbol{k}$ we have:

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 2 & 4 & 1 \\ -1 & 4 & -2 \end{vmatrix} = \boldsymbol{i}\begin{vmatrix} 4 & 1 \\ 4 & -2 \end{vmatrix} - \boldsymbol{j}\begin{vmatrix} 2 & 1 \\ -1 & -2 \end{vmatrix} + \boldsymbol{k}\begin{vmatrix} 2 & 4 \\ -1 & 4 \end{vmatrix}$$

$$= -12\boldsymbol{i} + 3\boldsymbol{j} + 12\boldsymbol{k}$$

which agrees with our previous answer.

**Example C.11.30**

Find $\boldsymbol{a} \times \boldsymbol{b}$ if $\boldsymbol{a} = 2\boldsymbol{i} + \boldsymbol{k}$ and $\boldsymbol{b} = 3\boldsymbol{i} - 4\boldsymbol{j} + 2\boldsymbol{k}$. Hence, find $|\boldsymbol{a} \times \boldsymbol{b}|$.

Using the determinant form of the cross product we have:

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 2 & 0 & 1 \\ 3 & -4 & 2 \end{vmatrix} = \boldsymbol{i}\begin{vmatrix} 0 & 1 \\ -4 & 2 \end{vmatrix} - \boldsymbol{j}\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} + \boldsymbol{k}\begin{vmatrix} 2 & 0 \\ 3 & -4 \end{vmatrix}$$

$$= (0 - (-4))\boldsymbol{i} - (4 - 3)\boldsymbol{j} + (-8 - 0)\boldsymbol{k}$$

$$= 4\boldsymbol{i} - \boldsymbol{j} - 8\boldsymbol{k}$$

Therefore, $|\boldsymbol{a} \times \boldsymbol{b}| = \sqrt{16 + 1 + 64} = \sqrt{81} = 9$

**Example C.11.31**

Find the angle between the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ if $\boldsymbol{a} = 2\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{b} = 3\boldsymbol{i} - 4\boldsymbol{j} + 2\boldsymbol{k}$.

We first need to determine $\boldsymbol{a} \times \boldsymbol{b}$:

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 2 & -1 & 1 \\ 3 & -4 & 2 \end{vmatrix} = \boldsymbol{i}\begin{vmatrix} -1 & 1 \\ -4 & 2 \end{vmatrix} - \boldsymbol{j}\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} + \boldsymbol{k}\begin{vmatrix} 2 & -1 \\ 3 & -4 \end{vmatrix}$$

$$= 2\boldsymbol{i} - \boldsymbol{j} - 5\boldsymbol{k}$$

Next, $|\boldsymbol{a} \times \boldsymbol{b}| = \sqrt{4+1+25} = \sqrt{30}$.

From $\boldsymbol{a} \times \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\sin\theta\hat{\boldsymbol{n}}$ we have that $|\boldsymbol{a} \times \boldsymbol{b}| = ||\boldsymbol{a}||\boldsymbol{b}|\sin\theta\hat{\boldsymbol{n}}|$ $= |\boldsymbol{a}||\boldsymbol{b}|\sin\theta$, where $\theta$ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$.

$|\boldsymbol{a}| = \sqrt{4+1+1} = \sqrt{6}$ and $|\boldsymbol{b}| = \sqrt{9+16+4} = \sqrt{29}$, so:

$$\sqrt{30} = \sqrt{6} \times \sqrt{29}\sin\theta \Leftrightarrow \sin\theta = \frac{\sqrt{30}}{\sqrt{6} \times \sqrt{29}}$$

$$\therefore \theta \approx 24°32'$$

Of course, it would have been much easier to do Example C.11.31 using the scalar product!

**Example C.11.32**

Find a vector of magnitude 5 units perpendicular to both $\boldsymbol{a} = -2\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{b} = \boldsymbol{i} - 3\boldsymbol{j} - \boldsymbol{k}$.

The cross product, $\boldsymbol{a} \times \boldsymbol{b}$, will provide a vector that is perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$. In fact, it is important to realise that the vector $\boldsymbol{a} \times \boldsymbol{b}$ is perpendicular to the plane that contains the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. This information will be very useful in the next sections, when the equation of a plane must be determined.

Let $\boldsymbol{c}$ be the vector perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$.

$\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b} =$

$$\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -2 & 1 & 1 \\ 1 & -3 & -1 \end{vmatrix} = \boldsymbol{i}\begin{vmatrix} 1 & 1 \\ -3 & -1 \end{vmatrix} - \boldsymbol{j}\begin{vmatrix} -2 & 1 \\ 1 & -1 \end{vmatrix} + \boldsymbol{k}\begin{vmatrix} -2 & 1 \\ 1 & -3 \end{vmatrix}$$

$$= (-1+3)\boldsymbol{i} - (2-1)\boldsymbol{j} + (6-1)\boldsymbol{k}$$

$$= 2\boldsymbol{i} - \boldsymbol{j} + 5\boldsymbol{k}$$

However, we want a vector of magnitude 5 units, that is, we want the vector $5\hat{\boldsymbol{c}}$.

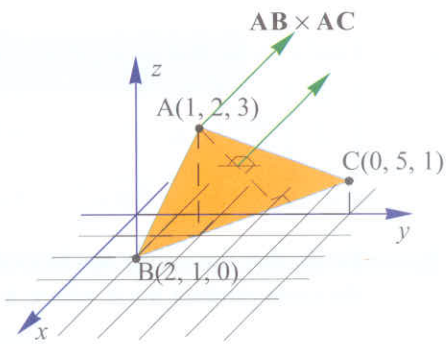
$$\hat{\boldsymbol{c}} = \frac{1}{|\boldsymbol{c}|}\boldsymbol{c} = \frac{1}{\sqrt{4+1+25}}(2\boldsymbol{i} - \boldsymbol{j} + 5\boldsymbol{k}) = \frac{1}{\sqrt{30}}(2\boldsymbol{i} - \boldsymbol{j} + 5\boldsymbol{k}).$$

So, $5\hat{\boldsymbol{c}} = \dfrac{5}{\sqrt{30}}(2\boldsymbol{i} - \boldsymbol{j} + 5\boldsymbol{k})$.

**Example C.11.33**

Find a unit vector that is perpendicular to the plane containing the points A(1, 2, 3), B(2, 1, 0) and C(0, 5, 1).

We start by drawing a diagram of the situation described so that the triangle ABC lies on the planes containing the points A, B and C.




Then, the vector, perpendicular to the plane containing the points A, B and C will be parallel to the vector produced by the cross product $\mathbf{AB} \times \mathbf{AC}$.

$$\text{Now, } \mathbf{AB} = \mathbf{AO} + \mathbf{OB} = -\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -3 \end{pmatrix}$$

$$\text{and } \quad \mathbf{AC} = \mathbf{AO} + \mathbf{OC} = -\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 0 \\ 5 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ -2 \end{pmatrix}.$$

Then,

$$\mathbf{AB} \times \mathbf{AC} = \begin{pmatrix} 1 \\ -1 \\ -3 \end{pmatrix} \times \begin{pmatrix} -1 \\ 3 \\ -2 \end{pmatrix} = \begin{pmatrix} -1 \times -2 - 3 \times -3 \\ -3 \times -1 - 1 \times -2 \\ 1 \times 3 - (-1) \times -1 \end{pmatrix} = \begin{pmatrix} 11 \\ 5 \\ 2 \end{pmatrix}$$

$$\text{Let } \boldsymbol{c} = \mathbf{AB} \times \mathbf{AC}, \therefore \hat{\boldsymbol{c}} = \frac{1}{\sqrt{150}}\begin{pmatrix} 11 \\ 5 \\ 2 \end{pmatrix}.$$

3-d realisation

### Exercise C.11.10

1. A set of vectors is defined by:

$$\boldsymbol{a} = \begin{pmatrix}1\\2\\3\end{pmatrix}, \boldsymbol{b} = \begin{pmatrix}-1\\2\\-3\end{pmatrix}, \boldsymbol{c} = \begin{pmatrix}1\\0\\5\end{pmatrix}, \boldsymbol{d} = \begin{pmatrix}-2\\-4\\3\end{pmatrix}$$

Find the vector products:

a $\boldsymbol{a} \times \boldsymbol{b}$ b $\boldsymbol{a} \times \boldsymbol{c}$ c $\boldsymbol{a} \times \boldsymbol{d}$

d $\boldsymbol{b} \times \boldsymbol{c}$ e $\boldsymbol{b} \times \boldsymbol{d}$ f $\boldsymbol{c} \times \boldsymbol{d}$

2. Find a vector that is perpendicular to both:

$$\begin{pmatrix}1\\2\\1\end{pmatrix} \text{ and } \begin{pmatrix}4\\6\\-2\end{pmatrix}.$$

3. Verify that the vector $\boldsymbol{a} = \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$ is perpendicular to the cross product $\boldsymbol{a} \times \boldsymbol{b}$ where $\boldsymbol{b} = 2\boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{k}$.

4. Verify that if $\boldsymbol{a} = \boldsymbol{i} + 6\boldsymbol{j} - 3\boldsymbol{k}$, $\boldsymbol{b} = -\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{c} = 2\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k}$ then:

a $\boldsymbol{a} \times (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \times \boldsymbol{b} + \boldsymbol{a} \times \boldsymbol{c}$.

b $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \bullet \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a} \bullet \boldsymbol{b})\boldsymbol{c}$.

5. If $\boldsymbol{a} = m\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$ and $\boldsymbol{b} = 2\boldsymbol{i} + n\boldsymbol{j} - \boldsymbol{k}$,

a Find: i $\boldsymbol{a} \times \boldsymbol{a}$ ii $\boldsymbol{a} \times \boldsymbol{b}$

b Show that $mn - 4 = 0$ if $\boldsymbol{a} \parallel \boldsymbol{b}$.

6. Find a vector that is perpendicular to both the vectors $\boldsymbol{i} + 6\boldsymbol{j} + 3\boldsymbol{k}$ and $\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$ and has a magnitude of 2.

7. Find a vector that is perpendicular to the plane containing the points:

a A(0, 0, 0), B(0, 5, 0) and C(2, 0, 0).

b A(2, 3, 1), B(2, 6, 2) and C(−1, 3, 4).

8. Using the cross product, find, to the nearest degree, the angle between the vectors:

a $\boldsymbol{u} = 2\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$ and $\boldsymbol{v} = -\boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k}$.

b $\boldsymbol{a} = 3\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$ and $\boldsymbol{b} = \boldsymbol{j} + \boldsymbol{k}$.

9. Prove that $(\boldsymbol{a} + \boldsymbol{b}) \times (\boldsymbol{a} - \boldsymbol{b}) = 2\boldsymbol{b} \times \boldsymbol{a}$.

10. Prove that $\boldsymbol{a} \bullet (\boldsymbol{a} \times \boldsymbol{b}) = \boldsymbol{b} \bullet (\boldsymbol{a} \times \boldsymbol{b}) = 0$.

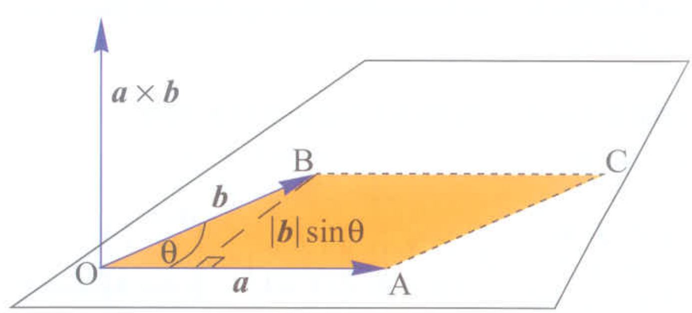
11. Prove that $|\boldsymbol{a} \times \boldsymbol{b}|^2 = |\boldsymbol{a}|^2|\boldsymbol{b}|^2 - (\boldsymbol{a} \bullet \boldsymbol{b})^2$.

12. What condition must the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ satisfy in order that the vectors $\boldsymbol{a} + \boldsymbol{b}$ and $\boldsymbol{a} - \boldsymbol{b}$ are collinear?

## Applications of the Vector Product

### 1. Area

Consider the parallelogram OACB lying on the plane, with the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ as shown.




Then, the area of OACB is given by:

$$\text{OA} \times |\boldsymbol{b}|\sin\theta = |\boldsymbol{a}|(|\boldsymbol{b}|\sin\theta) = |\boldsymbol{a}\times\boldsymbol{b}|$$

i.e. the **area of the parallelogram** OACB is given by the **magnitude of** the cross product $\boldsymbol{a}\times\boldsymbol{b}$.

We can prove this by using the result $|\boldsymbol{a}\times\boldsymbol{b}|^2 = |\boldsymbol{a}|^2|\boldsymbol{b}|^2-(\boldsymbol{a}\bullet\boldsymbol{b})^2$ where we replace $\boldsymbol{a}\bullet\boldsymbol{b}$ with $|\boldsymbol{a}||\boldsymbol{b}|\cos\theta$ and then carry through with some algebra. We leave this proof for the next set of exercises.

**Example C.11.34**

Find the area of the parallelogram determined by the vectors $\boldsymbol{a}=2\boldsymbol{i}+\boldsymbol{j}+3\boldsymbol{k}$ and $\boldsymbol{b}=\boldsymbol{i}+4\boldsymbol{j}-\boldsymbol{k}$.

We first need to determine the cross product, $\boldsymbol{a}\times\boldsymbol{b}$:

$$\boldsymbol{a}\times\boldsymbol{b}=\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 2 & 1 & 3 \\ 1 & 4 & -1 \end{vmatrix}=\begin{vmatrix} 1 & 3 \\ 4 & -1 \end{vmatrix}\boldsymbol{i}-\begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix}\boldsymbol{j}+\begin{vmatrix} 2 & 1 \\ 1 & 4 \end{vmatrix}\boldsymbol{k}$$

$$= -13\boldsymbol{i}+5\boldsymbol{j}+7\boldsymbol{k}$$

Now,

$$|\boldsymbol{a}\times\boldsymbol{b}| = |-13\boldsymbol{i}+5\boldsymbol{j}+7\boldsymbol{k}| = \sqrt{169+25+49} = \sqrt{243}$$

3-d realisation

Then, the area of the parallelogram is $\sqrt{243}$ unit$^2$.

**Example C.11.35**

Find the area of the triangle with vertices (1, 6, 3), (0, 10, 1) and (5, 8, 3).

We construct the vectors from the vertex (1, 6, 3) to the vertex (0, 10, 1) and also the vector from (1, 6, 3) to (5, 8, 3).

These vectors are:

$$\boldsymbol{a} = \begin{pmatrix} 0 \\ 10 \\ 1 \end{pmatrix}-\begin{pmatrix} 1 \\ 6 \\ 3 \end{pmatrix}=\begin{pmatrix} -1 \\ 4 \\ -2 \end{pmatrix}$$

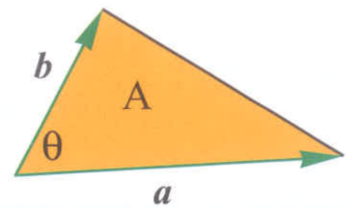
and

$$\boldsymbol{b} = \begin{pmatrix} 5 \\ 8 \\ 3 \end{pmatrix}-\begin{pmatrix} 1 \\ 6 \\ 3 \end{pmatrix}=\begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix}.$$

Next, we calculate the vector product:

$$\boldsymbol{a}\times\boldsymbol{b} = \begin{pmatrix} -1 \\ 4 \\ -2 \end{pmatrix}\times\begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix}=\begin{pmatrix} 4 \\ -8 \\ -18 \end{pmatrix}$$

Then, using the fact that $|\boldsymbol{a}\times\boldsymbol{b}| = |\boldsymbol{a}||\boldsymbol{b}|\sin\theta$ is a measure of the area of the parallelogram containing the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, we can deduce the **area**, $A$, **of the triangle** containing these vectors to be:




$$A = \frac{1}{2}|\boldsymbol{a}||\boldsymbol{b}|\sin\theta = \frac{1}{2}|\boldsymbol{a}\times\boldsymbol{b}|$$

In this case, the result is:

$$A=\frac{1}{2}\sqrt{4^2+(-8)^2+(-18)^2}=\frac{1}{2}\sqrt{404}\text{ units}^2=\sqrt{101}\text{ units}^2$$

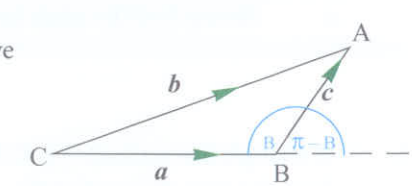
## 2. Geometric proofs

In the same way that we used the scalar product to neatly prove geometric theorems, for example, proving the cosine rule, we find that the vector product serves just as well for other geometric theorems. We now use the vector product to prove the sine rule.

**Example C.11.36**

By making use of the vector product, derive the sine rule, $\dfrac{a}{\sin A}=\dfrac{b}{\sin B}=\dfrac{c}{\sin C}$.

Consider the triangle ABC with associated vectors as shown:




From the diagram we have that $\boldsymbol{a} = \boldsymbol{b}-\boldsymbol{c}$.

Then:

$$\boldsymbol{a}\times\boldsymbol{a} = \boldsymbol{a}\times(\boldsymbol{b}-\boldsymbol{c})$$

But, as $\boldsymbol{a} \times \boldsymbol{a} = \boldsymbol{0}$,

then $\quad 0 = \boldsymbol{a} \times (\boldsymbol{b} - \boldsymbol{c})$

i.e. $\quad \boldsymbol{0} = \boldsymbol{a} \times \boldsymbol{b} - \boldsymbol{a} \times \boldsymbol{c}$

$$\Rightarrow \boldsymbol{a} \times \boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b}$$
$$\Rightarrow |\boldsymbol{a} \times \boldsymbol{c}| = |\boldsymbol{a} \times \boldsymbol{b}|$$

$\therefore |\boldsymbol{a}||\boldsymbol{c}|\sin(\pi - B) = |\boldsymbol{a}||\boldsymbol{b}|\sin C$

$\therefore |\boldsymbol{a}||\boldsymbol{c}|\sin B = |\boldsymbol{a}||\boldsymbol{b}|\sin C$ (as $\sin(\pi - B) = \sin B$ )

And so, $|\boldsymbol{c}|\sin B = |\boldsymbol{b}|\sin C$

$$\Leftrightarrow \frac{|\boldsymbol{c}|}{\sin C} = \frac{|\boldsymbol{b}|}{\sin B}$$

That is, $\dfrac{c}{\sin C} = \dfrac{b}{\sin B}$.

Similarly, we can prove that $\dfrac{|\boldsymbol{c}|}{\sin C} = \dfrac{|\boldsymbol{a}|}{\sin A}$, leading to the results:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

## Exercise C.11.11

1. Find the area of the parallelogram with adjacent vectors:

   a $\quad 2\boldsymbol{i} + \boldsymbol{k}$ and $-\boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k}$

   b $\quad 3\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$ and $5\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$

2. A parallelogram has two adjacent sides formed by the vectors:
   $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ and $\dfrac{1}{2}\begin{pmatrix} -14 \\ 1 \\ -1 \end{pmatrix}$.

   a $\quad$ Find the cross product of these two vectors.

   b $\quad$ Find the area of this parallelogram.

   c $\quad$ Hence find the angle between the two vectors.

3. A triangle has vertices (–1, 2, 4), (3, 7, –5) and (4, 2, 3). Find the area of this triangle.

4. Find $x$, where $x > 0$, if the area of the triangle formed by the adjacent vectors $x\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$ and $\boldsymbol{j} - \boldsymbol{k}$ is 12 unit$^2$.

5. Find the area of the triangle with adjacent sides formed by the vectors $2\boldsymbol{i} + 3\boldsymbol{j} - 4\boldsymbol{k}$ and $2\boldsymbol{i} - 3\boldsymbol{j} + 4\boldsymbol{k}$. Hence find the angle enclosed by these two vectors.

6. Show that the quadrilateral with vertices at O(4, 1, 0), A(7, 6, 2), B(5, 5, 4) and C(2, 0, 2) is a parallelogram. Hence find its area.

7. Find the area of the parallelogram having diagonals $\boldsymbol{u} = 3\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$ and $\boldsymbol{v} = \boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}$.

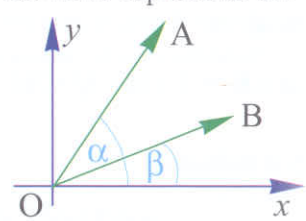
8. If $\boldsymbol{a}$ and $\boldsymbol{b}$ are three-dimensional vectors and $\theta$ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$, use the result that $|\boldsymbol{a} \times \boldsymbol{b}|^2 = |\boldsymbol{a}|^2|\boldsymbol{b}|^2 - (\boldsymbol{a} \bullet \boldsymbol{b})^2$ to prove that $|\boldsymbol{a} \times \boldsymbol{b}| = |\boldsymbol{a}||\boldsymbol{b}|\sin\theta$.

9. Find, in terms of $\alpha$ and $\beta$ the vector expressions for:

   a i $\quad$ **OA**

   ii $\quad$ **OB**

   where both **OA** and **OB** are unit vectors.




   b $\quad$ Use the vector product to prove the trigonometric identity $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \sin\beta\cos\alpha$.

10. Let ABCD be a quadrilateral such that its diagonals, [AC] and [BD], intersect at some point O. If triangle ABC has the same area as triangle CBD, show that O is the mid-point of the diagonal [AC].

11. Show that the condition for three points A, B and C to be collinear is that their respective position vectors, $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ satisfy the equation $(\boldsymbol{a} \times \boldsymbol{b}) + (\boldsymbol{b} \times \boldsymbol{c}) + (\boldsymbol{c} \times \boldsymbol{a}) = \boldsymbol{0}$.

12. Prove that the volume of the parallelepiped determined by the vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ is given by $|\boldsymbol{a} \bullet (\boldsymbol{b} \times \boldsymbol{c})|$.


Find the volume of the parallelepiped determined by the vectors:

$$\boldsymbol{a} = 2\boldsymbol{i}+\boldsymbol{j}+3\boldsymbol{k}\ ,\ \boldsymbol{b} = \boldsymbol{i}+4\boldsymbol{j}-\boldsymbol{k} \text{ and } \boldsymbol{c} = -2\boldsymbol{i}+\boldsymbol{j}+5\boldsymbol{k}$$

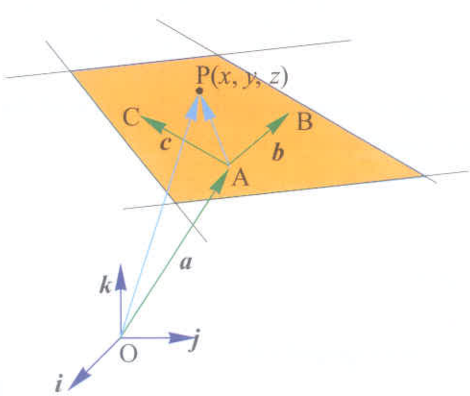
13. a Consider the triangle ABC where the points M, N and P lie on the sides [AB], [BC] and [CA] respectively and are such that $\mathbf{AM} = k_1\mathbf{AB}$, $\mathbf{BN} = k_2\mathbf{BC}$ and $\mathbf{CP} = k_3\mathbf{CA}$, where $k_1, k_2, k_3 \in \mathbb{R}$. Show that if the vectors **CM**, **AN** and **BP** form a triangle, then, $k_1 = k_2 = k_3$.

b Consider the triangle ABC where the points M, N and P lie on the sides [AB], [BC] and [CA] respectively and are such that $\mathbf{AM} = k\mathbf{AB}$, $\mathbf{BN} = k\mathbf{BC}$ and $\mathbf{CP} = k\mathbf{CA}$, and $k \in \mathbb{R}$. Find the value of $k$ so that the area of the triangle formed by the vectors **CM**, **AN** and **BP** is a minimum.

## Vector Equation of a Plane

The approach to determine the vector equation of a plane requires only a small extension of the ideas of the previous sections. In fact, apart from introducing the form that the equation of a plane has, this section has its foundations in our most recent work.

We begin with the vector equation of a plane.




Let P(x, y, z), whose position vector is $\boldsymbol{r} = \mathbf{OP}$ be any point on the plane relative to some origin O.

Consider three points, A, B and C on this plane where $\mathbf{OA} = \boldsymbol{a}$, $\mathbf{AB} = \boldsymbol{b}$ and $\mathbf{AC} = \boldsymbol{c}$. That is, the plane contains the vectors $\boldsymbol{b}$ and $\boldsymbol{c}$, where $\boldsymbol{b}$ ¦ **0** ¦ $\boldsymbol{c}$ and the vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are non-coplanar.

Now, as **AP**, $\boldsymbol{b}$ and $\boldsymbol{c}$ are coplanar, then we can express **AP** in terms of $\boldsymbol{b}$ and $\boldsymbol{c}$: $\mathbf{AP} = \lambda\boldsymbol{b} + \mu\boldsymbol{c}$ for some real $\lambda$ and $\mu$.

Then, $\boldsymbol{r} = \mathbf{OP} = \mathbf{OA} + \mathbf{AP} = \boldsymbol{a} + \lambda\boldsymbol{b} + \mu\boldsymbol{c}$.

That is, every point on the plane has a position vector of this form.

As such, we say that the vector equation of a plane is given by $\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{b} + \mu\boldsymbol{c}$

This means that to find the vector form of the equation of a plane we need to know:

1. the position vector of a point A in the plane, and
2. two non-parallel vectors in the plane.

**Example C.11.37**

Find the vector equation of the plane containing the vectors:

$\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}$ which also includes the point (1, 2, 0).

Let $\boldsymbol{b} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ and $\boldsymbol{c} = \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}$ be two vectors on the plane.

Then, as the point (1, 2,0) lies on the plane we let $\boldsymbol{a} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ be the position of this point.

Using the vector form of the equation of a plane,

i.e. $\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{b} + \mu\boldsymbol{c}$, we have $\boldsymbol{r} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + \mu\begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}$.-

## Cartesian Equation of a Plane

In the same way that we were able to produce a Cartesian equation for a line in 2-D, we now derive the Cartesian equation of a plane.

Using Example C.11.37 we obtain the parametric equations and use them to derive the Cartesian equation of the plane.

From the vector equation $\boldsymbol{r} = \begin{pmatrix}1\\2\\0\end{pmatrix} + \lambda\begin{pmatrix}2\\1\\1\end{pmatrix} + \mu\begin{pmatrix}3\\0\\-1\end{pmatrix}$ we obtain the following parametric equations:

$x = 1 + 2\lambda + 3\mu$ – (1)

$y = 2 + \lambda$......– (2)

$z = \lambda - \mu$ – (3)

Now we find expressions for $\lambda$ and $\mu$ in terms of $x$, $y$ and $z$, taking care to use all three equations while doing this:

From (1) and (3) we obtain: $\lambda = \dfrac{x + 3z - 1}{5}$ – (4)

From (2) and (3) we obtain: $\mu = y - z - 2$ – (5)

Finally we substitute these back into one of the equations. In this particular case it will be easiest to use (4) and (2) - and in fact we didn't need the expression for $\mu$, though in most cases we will.

Substituting (4) into (2) we obtain: $y = 2 + \dfrac{x + 3z - 1}{5}$

and simplifying we get: $x - 5y + 3z = -9$ .

This result tells us that the:

> Cartesian form of a plane is given by the equation:
>
> $ax + by + cz = d$

### Example C.11.38

Find the Cartesian equation of the plane defined by the vector equation:

$$\boldsymbol{r} = \begin{pmatrix}1\\3\\4\end{pmatrix} + \lambda\begin{pmatrix}-2\\1\\1\end{pmatrix} + \mu\begin{pmatrix}1\\1\\2\end{pmatrix}$$

From the vector equation of the plane, namely:

$$\boldsymbol{r} = \begin{pmatrix}1\\3\\4\end{pmatrix} + \lambda\begin{pmatrix}-2\\1\\1\end{pmatrix} + \mu\begin{pmatrix}1\\1\\2\end{pmatrix},$$

we produce the parametric equations:

$x = 1 - 2\lambda + \mu$ – (1)

$y = 3 + \lambda + \mu$ – (2)

$z = 4 + \lambda + 2\mu$ – (3)

Next, we eliminate $\lambda$ and $\mu$ :(2) – (1): $y - x = 2 + 3\lambda$ – (4)

2 × (2) – (3): $2y - z = 2 + \lambda$ – (5)

(4) – 3 × (5): $-5y - x + 3z = -4$

That is, the Cartesian equation of the plane is given by $-5y - x + 3z = -4$ or $x + 5y - 3z = 4$ .

### Exercise C.11.12

1. Find the vector equation of the plane containing the vectors $\boldsymbol{b}$ and $\boldsymbol{c}$ and passing through the point $A$. In each case, draw a rough diagram depicting the situation.

   a $\boldsymbol{b} = 3\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}$ , $\boldsymbol{c} = -2\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$ , $A \equiv (1, 0, 1)$ .

   b $\boldsymbol{b} = \boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$ , $\boldsymbol{c} = -\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$ , $A \equiv (-1, 2, 1)$ .

   c $\boldsymbol{b} = 2\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$, $\boldsymbol{c} = 2\boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k}$ , $A \equiv (4, 1, 5)$ .

   d $\boldsymbol{b} = -3\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}$ , $\boldsymbol{c} = \boldsymbol{i} - 2\boldsymbol{j} + \frac{1}{2}\boldsymbol{k}$ , $A \equiv (2, -3, -1)$ .

2. Find the Cartesian equation for each of the planes in Question **1**.

3. Find the:

   i vector equation.

   ii Cartesian equation of the plane containing the points:

   a A(2, 3, 4), B(–1, 2, 1) and C(0, 5, 6).

   b A(3, –1, 5), B(1, 4, –6) and C(2, 3, 4).

4. A plane contains the vectors $\boldsymbol{b} = 2\boldsymbol{i}-\boldsymbol{j}-\boldsymbol{k}$ and $\boldsymbol{c} = 3\boldsymbol{i}+\boldsymbol{j}+2\boldsymbol{k}$.

   a Find the vector equation of the plane, containing the vectors $\boldsymbol{b}$ and $\boldsymbol{c}$ and passing through the point:

   i (2, −2, 3).

   ii (0, 0, 0).

   b Find the Cartesian equation for each plane in part **a**.

   c Express $\boldsymbol{b}\times\boldsymbol{c}$ in the form $a\boldsymbol{i}+b\boldsymbol{j}+c\boldsymbol{k}$.

   d What do you notice about the coefficient of $x$, $y$ and $z$ in part **b** and the values $a$, $b$ and $c$ from part **c**?

## Normal Vector Form of a Plane

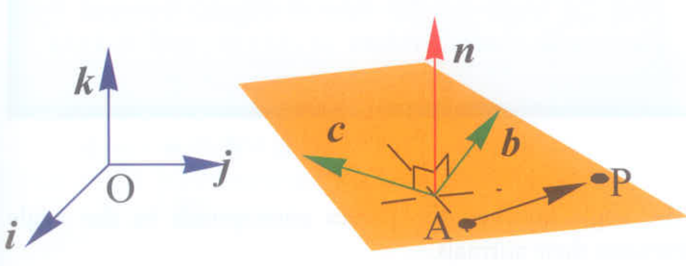
Before we formally derive the **normal vector form of a plane**, we consider an example that follows directly from the work covered so far. In particular, Question **4** from Exercise C.11.12 – if you have not attempted this problem you should do so now, before proceeding further.

Consider a plane containing the vectors $\boldsymbol{b} = 3\boldsymbol{i}-\boldsymbol{j}+2\boldsymbol{k}$ and $\boldsymbol{c} = 2\boldsymbol{i}+2\boldsymbol{j}+\boldsymbol{k}$ and passing through the point A(2, 1, 6). Now, the cross product $\boldsymbol{b}\times\boldsymbol{c}$ represents a vector that is perpendicular to the plane containing the vectors $\boldsymbol{b}$ and $\boldsymbol{c}$.

$$\text{Let } \boldsymbol{n} = \boldsymbol{b}\times\boldsymbol{c} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 3 & -1 & 2 \\ 2 & 2 & 1 \end{vmatrix} = \begin{vmatrix} -1 & 2 \\ 2 & 1 \end{vmatrix}\boldsymbol{i} - \begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix}\boldsymbol{j} + \begin{vmatrix} 3 & -1 \\ 2 & 2 \end{vmatrix}\boldsymbol{k}$$

$$= -5\boldsymbol{i}+\boldsymbol{j}+8\boldsymbol{k}$$

We now have a vector, $\boldsymbol{n} = -5\boldsymbol{i}+\boldsymbol{j}+8\boldsymbol{k}$ that is perpendicular to the plane in question.




3-d realisation - plane and perpendicular vector

Next, consider any point P(x, y, z) on this plane. As P lies on the plane the vector **AP** must also be perpendicular to the vector $\boldsymbol{n}$. This means that $\boldsymbol{n} \bullet \mathbf{AP} = 0$.

To use the equation $\boldsymbol{n} \bullet \mathbf{AP} = 0$ we first need to find the vector **AP**. As **AP** = **AO** + **OP**, we have:

$$\begin{aligned} \mathbf{AP} &= -(2\boldsymbol{i}+\boldsymbol{j}+6\boldsymbol{k})+(x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k}) \\ &= (x-2)\boldsymbol{i}+(y-1)\boldsymbol{j}+(z-6)\boldsymbol{k} \end{aligned}$$

Then, from $\boldsymbol{n} \bullet \mathbf{AP} = 0$ we have

$$\begin{aligned} (-5\boldsymbol{i}+\boldsymbol{j}+8\boldsymbol{k}) \bullet ((x-2)\boldsymbol{i}+(y-1)\boldsymbol{j}+(z-6)\boldsymbol{k}) &= 0 \\ \Leftrightarrow -5(x-2)+(y-1)+8(z-6) &= 0 \\ \Leftrightarrow -5x+y+8z &= 39 \end{aligned}$$

That is, we have obtained the Cartesian equation of the plane containing the vectors $\boldsymbol{b} = 3\boldsymbol{i}-\boldsymbol{j}+2\boldsymbol{k}$ and $\boldsymbol{c} = 2\boldsymbol{i}+2\boldsymbol{j}+\boldsymbol{k}$ and passing through the point A(2, 1, 6) without making use of the parametric form of the plane.

We check this result using the parametric form of the plane.

From the vector for, $\boldsymbol{r} = \begin{pmatrix} 2 \\ 1 \\ 6 \end{pmatrix} + \lambda\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} + \mu\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$

we obtain the parametric equations:

$x = 2+3\lambda+2\mu$ – (1)

$y = 1-\lambda+2\mu$ – (2)

and $z = 6+2\lambda+\mu$ – (3)

(1) – (2): $x-y = 1+4\lambda$ – (4)

(2) – 2 × (3): $y-2z = -11-5\lambda$ – (5)

From (4) and (5) we obtain:

$$\frac{x-y-1}{4} = \frac{y-2z+11}{-5} \Leftrightarrow -5x+y+8z = 39.$$

As expected, we produce the same equation.

To use this method, we require a vector that is perpendicular to the plane and a point that lies on the plane. We could use the vector, $\boldsymbol{n}$ (say) or the unit vector $\hat{\boldsymbol{n}}$, or even $-\boldsymbol{n}$, as they are all perpendicular to the plane.

We can summarise this process as follows:

To find the Cartesian equation of a plane through the point $P_0(x_0, y_0, z_0)$ having a non-zero normal vector $\boldsymbol{n}$ (or $\hat{\boldsymbol{n}}$) we

1. let $P(x, y, z)$ be any point on the plane, and

2. find the vector $\boldsymbol{n} = a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k}$ .

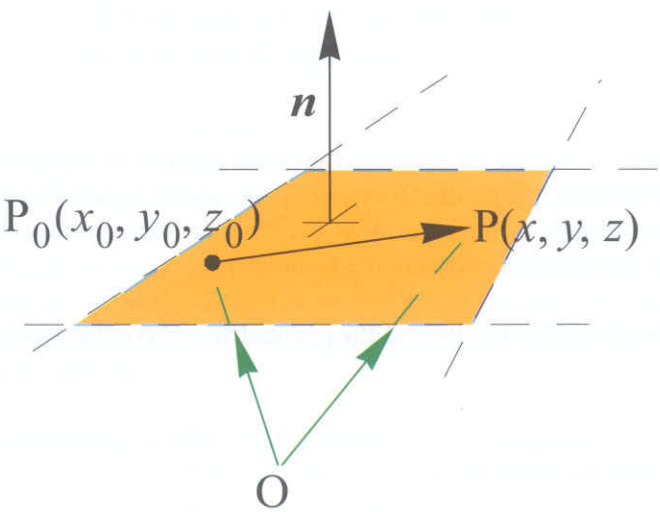
Then, as $\mathbf{P_0P} \perp \boldsymbol{n}$ for all points P on the plane, we have

$$\mathbf{P_0P} \bullet \boldsymbol{n} = 0$$

$$\Rightarrow [(x-x_0)\boldsymbol{i} + (y-y_0)\boldsymbol{j} + (z-z_0)\boldsymbol{k}] \bullet (a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k}) = 0$$

$$\therefore a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$$

Or, after some simplifying, $ax + by + cz = d$




Notice that if two planes, $\Pi_1$ and $\Pi_2$ have normal vectors, $\boldsymbol{n}_1 = a_1\boldsymbol{i} + b_1\boldsymbol{j} + c_1\boldsymbol{k}$ and $\boldsymbol{n}_2 = a_2\boldsymbol{i} + b_2\boldsymbol{j} + c_2\boldsymbol{k}$ respectively, then the two planes, $\Pi_1$ and $\Pi_2$ are:

1. parallel iff their normal vectors are parallel,

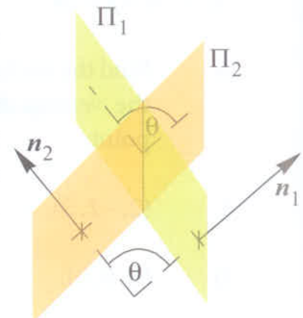
   i.e. iff $\boldsymbol{n}_1 = m \times \boldsymbol{n}_2$, where $m \in \mathbb{R}$

   i.e. iff $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2} = m$

2. perpendicular iff their normal vectors are perpendicular. i.e. iff $\boldsymbol{n}_1 \bullet \boldsymbol{n}_2 = 0$

   i.e. iff $a_1a_2 + b_1b_2 + c_1c_2 = 0$

Taking this one step further, this result also means that we can use the normals to find the angle between two planes. The angle between two planes is defined as the angle between their normals.




If two planes, $\Pi_1$ and $\Pi_2$ have normal vectors $\boldsymbol{n}_1 = a_1\boldsymbol{i} + b_1\boldsymbol{j} + c_1\boldsymbol{k}$ and $\boldsymbol{n}_2 = a_2\boldsymbol{i} + b_2\boldsymbol{j} + c_2\boldsymbol{k}$ respectively, and intersect at an acute angle $\theta$ (or $\pi - \theta$ depending on their direction), the acute angle $\theta$ can be found from the product rule:

$$\cos\theta = \frac{|\boldsymbol{n}_1 \bullet \boldsymbol{n}_2|}{|\boldsymbol{n}_1||\boldsymbol{n}_2|}$$

**Example C.11.39**

Find the Cartesian equation of the plane containing the point A(3, 1, 1) and with the normal vector given by $\boldsymbol{n} = 3\boldsymbol{i} - 2\boldsymbol{j} + 4\boldsymbol{k}$.

Using the normal vector, $\boldsymbol{n} = 3\boldsymbol{i} - 2\boldsymbol{j} + 4\boldsymbol{k}$ and a vector on the plane passing through the point A(3, 1, 1), i.e. the vector $\mathbf{AP} = (x-3)\boldsymbol{i} + (y-1)\boldsymbol{j} + (z-1)\boldsymbol{k}$ , where $P(x, y, z)$ is an arbitrary point on the plane, we have

$$\boldsymbol{n} \bullet \mathbf{AP} = 0$$

$$\Rightarrow (3\boldsymbol{i} - 2\boldsymbol{j} + 4\boldsymbol{k}) \bullet [(x-3)\boldsymbol{i} + (y-1)\boldsymbol{j} + (z-1)\boldsymbol{k}] = 0$$

That is, $3(x-3) + (-2)(y-1) + 4(z-1) = 0$

Or, after some simplification, $3x - 2y + 4z = 11$

**Example C.11.40**

Find the angle (to the nearest degree) between the planes with normal vectors $3\boldsymbol{i} - 2\boldsymbol{j} + 4\boldsymbol{k}$ and $\boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k}$ .

The angle between the planes corresponds to the angle between their normals.

So, using the dot product we have

$$(3i-2j+4k)\bullet(i-j+3k) = |3i-2j+4k||i-j+3k|\cos\theta$$
$$3+2+12 = \sqrt{29}\times\sqrt{11}\cos\theta$$
$$\therefore \cos\theta = \frac{17}{\sqrt{29}\times\sqrt{11}}$$

And so, we have that $\theta \approx 17°52' = 18°$ (to the nearest degree).

**Example C.11.41**

Find the angle (to the nearest degree) between the planes $2x + 3y - 8z = 9$ and $-x + y - 2z = 1$.

To find the angle between the planes we need the normal vectors to the planes. From our observations, we have that a normal vector can be directly obtained from the equation of a plane by using the coefficients of each variable.

For the plane $2x + 3y - 8z = 9$, a normal vector would be $2i + 3j - 8k$ and for the plane $-x + y - 2z = 1$, a normal vector would be $-i+j-2k$.

Then, we proceed as in Example 4.6.4, using the cosine rule:

$$(2i+3j-8k)\bullet(-i+j-2k)=|2i+3j-8k||-i+j-2k|\cos\theta$$
$$\therefore -2+3+16 = \sqrt{77}\times\sqrt{6}\cos\theta$$
$$\therefore \cos\theta = \frac{17}{\sqrt{77}\times\sqrt{6}}$$

That is, $\theta \approx 37°44' = 38°$ (to the nearest degree).

## Exercise C.11.13

1. Find the Cartesian equation of the plane containing the point P and having a normal vector, $\boldsymbol{n}$.

   a $\boldsymbol{n} = 2i-j+5k$, $P \equiv (3, 4, 1)$

   b $\boldsymbol{n} = -4i+6j-8k$, $P \equiv (-2, 3, -1)$

   c $|\boldsymbol{a}| = 7, |\boldsymbol{b}| = 3$, $P \equiv (2, 4, 5)$

   d $\boldsymbol{n} = 5i+2j+k$, $P \equiv (-1, 2, 1)$

2. Which of the planes in Question 1 pass through the origin?

3. Find the Cartesian equation of the plane containing the points:

   a A(2, 1, 5), B(3, 2, 7) and C(0, 1, 2)

   b A(0, 2, 4), B(1, 2, 3) and C(4, 2, 5)

   c A(1, 1, 7), B(2, –1, 5) and C(–1, 3, 7)

4. Find the angle (to the nearest degree) between the planes with normal vectors:

   a $i-j+k$ and $i-j+3k$.

   b $-3i+5j-2k$ and $j+k$.

   c $4i-2j+7k$ and $2i+11j+2k$.

   d $-3i+2j-4k$ and $9i-6j+8k$.

5. Find the angle between the planes:

   a $\Pi_1 : -x+3y-z = 9$ and $\Pi_2 : 6x+2y+3z = 4$

   b $\Pi_1 : 2x+2y-3 = z$ and $\Pi_2 : 2y-3z+2 = 0$

   c $\Pi_1 : 2x-y+3z = 2$ and $\Pi_2 : 2x+y-7z = 8$

6. Find the equation of the plane which passes through the point A(4, 2, 1) and:

   a contains the vector joining the points B(3, –2, 4) and C(5, 0, 1).

   b is perpendicular to the planes with equations $5x-2y+6z+1=0$ and $2x-y-z=4$.

7. Find the equation of the plane which passes through the point A(–1, 2, 1) and is parallel to the plane $x-2y+3z+2 = 0$.

8. Find the equation of the plane which passes through the point A(−1, 2, 1) and is parallel to the plane $2y - 3 = 3x + 5z$.

9. The planes $4x - y + 6z = -5$ and $ax + by - z = 7$ are perpendicular. If both planes contain the point (1, 3, −1), find $a$ and $b$.

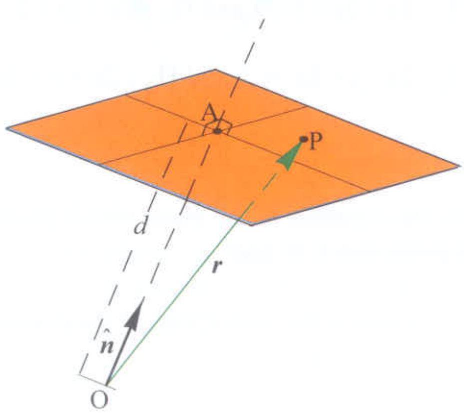
10. a Find a vector equation of the line passing through the points (3, 2, 1) and (5, 7, 6).

b Find the normal vector of the plane $3x + 2y + z = 6$.

c Hence, find the inclination that the line $\frac{x-3}{2} = \frac{y-2}{5} = \frac{z-1}{5}$ makes with the plane $3x + 2y + z = 10$.

## The Normal Form

We now formalise (or at least give a complete vectorial presentation for) the equation of a plane in three dimensions. The good news is that the normal form of the vector equation of a plane in three dimensions develops in almost the same way as the vector equation of a line in two and three dimensions.




Let $\hat{n}$ be a (unit) vector from O normal to the plane and $d$ be the distance of the plane from the origin.

The condition for a point P to be on the plane is that **OA** is perpendicular to **AP**.

That is, $\mathbf{OA} \bullet \mathbf{AP} = 0$

Now, $\mathbf{AP} = \mathbf{AO} + \mathbf{OP} = -d\hat{n} + \boldsymbol{r}$

So that $d\hat{n} \bullet (-d\hat{n} + \boldsymbol{r}) = 0$

Now, dividing by $d$ (assumed to be non-zero)

we have: $\hat{n} \bullet (-d\hat{n} + \boldsymbol{r}) = 0$

$$\therefore -d\hat{n} \bullet \hat{n} + \hat{n} \bullet \boldsymbol{r} = 0$$
$$\Rightarrow \hat{n} \bullet \boldsymbol{r} = d\hat{n} \bullet \hat{n}$$
$$\therefore \hat{n} \bullet \boldsymbol{r} = d \text{ (as } \hat{n} \bullet \hat{n} = 1\text{)}$$

That is, the normal vector form of the equation of a plane is given by $\hat{n} \bullet \boldsymbol{r} = d$.

If we are using $\boldsymbol{n}$ (not a unit vector) the equation becomes $\boldsymbol{n} \bullet \boldsymbol{r} = D$, where $D$ is no longer the distance of the plane from the origin.

If we know the position vector $\boldsymbol{a}$ of a point on the plane we can write the equation as:

$$\boldsymbol{r} \bullet \boldsymbol{n} = \boldsymbol{a} \bullet \boldsymbol{n}$$

For example $\boldsymbol{r} \bullet \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 8$ is the equation of a plane.

We can get this into a Cartesian form by noting that $\boldsymbol{r}$ is the position vector of some arbitrary point P($x$, $y$, $z$) on the plane and so we can write the vector expression as:

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 8$, or $x + y + z = 8$.

Converting from Cartesian to vector form:

$2x - y + 4z = 2$ becomes $\boldsymbol{r} \bullet \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} = 2$.

If we want to get the equation in $\hat{n}$ form, i.e. in the form $\hat{n} \bullet \boldsymbol{r} = d$ we can work out that the length of the vector $\begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}$ is $\sqrt{2^2 + (-1)^2 + 4^2} = \sqrt{21}$, and so, from the equation

$\boldsymbol{r} \bullet \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} = 2$ we divide both sides by $\sqrt{21}$ to get:

$$\frac{1}{\sqrt{21}}\boldsymbol{r} \bullet \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} = \frac{1}{\sqrt{21}} \times 2 \text{ or } \boldsymbol{r} \bullet \frac{1}{\sqrt{21}}\begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} = \frac{2}{\sqrt{21}}.$$

The distance of the plane from the origin is $\frac{2}{\sqrt{21}}$.

**Example C.11.42**

Show that the line $r = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + k\begin{pmatrix} 4 \\ 3 \\ 1 \end{pmatrix}$ is parallel to the plane $x - 2y + 2z = 11$.

We need to prove that $\boldsymbol{n}$ is perpendicular to $\boldsymbol{v}$.

Rewriting $x - 2y + 2z = 11$ in the normal vector form, we have:

$$r \bullet \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} = 11$$

From this equation, a suitable $\boldsymbol{n}$ is the vector $\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$.

From the vector equation of the line, the direction vector of $\boldsymbol{v}$ is:

$$\begin{pmatrix} 4 \\ 3 \\ 1 \end{pmatrix}$$

As $\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} 4 \\ 3 \\ 1 \end{pmatrix} = 4 - 6 + 2 = 0$, the vectors are perpendicular.

So the line and plane are parallel.

**Example C.11.43**

Show that the line: $r = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + s\begin{pmatrix} 5 \\ 1 \\ -1 \end{pmatrix}$

lies in the plane $x - 3y + 2z = -1$.

From the vector equation of the line we obtain the parametric equations:

$$x = 2 + 5s$$

$$y = 1 + s$$

and $$z = -s$$

If this line lies on the plane, then the parametric equations must satisfy the Cartesian equation of the plane. Substituting, into the equation $x - 3y + 2z = -1$, we get

$$\text{L.H.S} = x - 3y + 2z = (2 + 5s) - 3(1 + s) - 2s$$
$$= 2 + 5s - 3 - 3s - 2s$$
$$= -1$$

= R.H.S - Therefore, the line lies in the plane.

## Exercise C.11.14

For this set of exercises, where appropriate, make use of the normal vector form to solve the questions.

1. Convert these planes to Cartesian and vector form:

   a $r = \begin{pmatrix} 1 \\ 1 \\ -4 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix} + \mu\begin{pmatrix} -3 \\ 0 \\ -1 \end{pmatrix}$

   b $r = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + \mu\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

2. Given A(1, 1, 0), B(2, 1, 3) and C(1, 2, −1), find the Cartesian equation of the plane containing A, B and C. (Find a parametric form first by taking A as the point in the plane and **AB** and **AC** as the two vectors in the plane.)

3. Re-solve Question 2 by taking the Cartesian form as $x + by + cz = d$, then calculating $b$, $c$ and $d$ (simultaneous equations in three unknowns).

4. Show that the line $x + 1 = \frac{y+2}{3} = \frac{4-z}{4}$ and the plane $5x + y + 2z = 20$ are parallel.

5. Find the distance of each of these planes from the origin (i.e. find $d$):

   a $2x - 3y + 6z = 21$

   b $2x - y + 2z = 5$

   c $x + y - 3z = 11$

   d $4x + 2y - z = 20$

6. Find the equation of the plane through (1, 2, 3) parallel to $3x + 4y - 5z = 0$.

7. Find the equation of the plane through the three points (1, 1, 0), (1, 2, 1) and (−2, 2, −1).

8. Show that the four points (0, −1, 0), (2, 1, 1), (1, 1, 1) and (3, 3, 2) are coplanar.

9. Find the equation of the plane through (2, −3, 1) normal to the line joining (3, 4, −1) and (2, −1, 5).

## Intersection of Two Lines

In general, two lines (in three dimensions) will not intersect, but in certain circumstances they may. We can show, for example, that the lines:

$$\boldsymbol{r} = \begin{pmatrix}-1\\4\\0\end{pmatrix} + \lambda\begin{pmatrix}3\\-2\\1\end{pmatrix} \text{ and } \boldsymbol{r} = \begin{pmatrix}4\\4\\-1\end{pmatrix} + \mu\begin{pmatrix}2\\-3\\2\end{pmatrix}$$

*do* intersect, and we can find their point of intersection.

We show that there exist values of $\lambda$ and $\mu$ which make the $x$-, $y$- and $z$- coordinates of the two lines identical. If we compare the $x$- and $y$-coordinates we get:

$$\mathbf{AP} = \lambda\boldsymbol{b} + \mu\boldsymbol{c}$$

We can solve these to get $\lambda = 3$ and $\mu = 2$. The point that will decide whether the two lines intersect is:

when $\lambda = 3$ and $\mu = 2$, are the $z$-coordinates also equal?

This can be tested: $\lambda = 3$ and $\mu = 2$, $l$ has $z$-coordinate $= 0 + \lambda = 3$ and $m$ has $z$-coordinate $= -1 + 2\mu = 3$. So the lines do intersect.

Substituting $\lambda = 3$ and $\mu = 2$ in the expressions for the $x$- and $y$-coordinates we find that the point of intersection is (8, −2, 3). If the $z$-coordinates had been different, we would deduce that the lines do not intersect.

Recall that lines which do not intersect and are not parallel (a situation we looked at in section 4.4) are said to be ***skew***.

### Exercise 4.7.1

1. a Show that the lines $r_A = 5i + j + k + \lambda(i + 2j - 2k)$ and $r_B = 11i + 4j - 2k + \mu(4i - j + k)$ intersect, and find their point of intersection.

   b By considering the scalar product $(\boldsymbol{i} + 2\boldsymbol{j} - 2\boldsymbol{k}) \bullet (4\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k})$, show that the lines from part **a** intersect at right angles.

2. Given the lines:

   a $\boldsymbol{r} = \begin{pmatrix}5\\2\\3\end{pmatrix} + \kappa\begin{pmatrix}-1\\3\\5\end{pmatrix}$

   b $\boldsymbol{r} = \begin{pmatrix}1\\2\\1\end{pmatrix} + \lambda\begin{pmatrix}2\\-2\\3\end{pmatrix}$

   c $\boldsymbol{r} = \begin{pmatrix}5\\-6\\2\end{pmatrix} + \mu\begin{pmatrix}1\\1\\4\end{pmatrix}$,

   find the two lines that intersect. Find also the coordinates of the point of intersection and the acute angle between the two lines.

3. Show that the line joining (1, 4, 3) to (7, −5, −6) intersects the line

$$\frac{x-1}{2} = -y = \frac{3-z}{3},$$

   and find the point of intersection. (Find a parametric form for each line – remember to use a different parameter for each line.)

4. Show that the three lines:

   L: $x = y + 4 = \frac{z}{2} + 1$ M: $\frac{x-1}{3} = 2y + 1 = z - 5$

   N: $\frac{x}{4} = y + 1 = \frac{z-3}{3}$

   intersect at a single point, and give its coordinates.

## Intersection of a Line and a Plane

We have considered the case of a line and a plane being parallel, and the case of a line lying in a plane. If neither of these happens then the line and plane must intersect in a point.

The angle between a line and a plane is defined as the angle between the line and its projection on the plane. To find the angle between a line and a plane we look at the vectors $\boldsymbol{n}$ (perpendicular to the plane) and $\boldsymbol{v}$ (in the direction of the line):

We can find angle $\phi$ from the formula $\cos\phi = \frac{\boldsymbol{v} \bullet \boldsymbol{n}}{|\boldsymbol{v}||\boldsymbol{n}|}$ then subtract from 90° to find $\theta$.

Alternatively we can use the fact that $\cos\phi = \sin\theta$ to write directly $\sin\theta = \frac{\boldsymbol{v} \bullet \boldsymbol{n}}{|\boldsymbol{v}||\boldsymbol{n}|}$.

### Example C.11.44

Find the point of intersection of the line

$\frac{x}{2} = \frac{y+6}{2} = 3z - 1$ and the plane $3x + y - z = 9$.

Find also the angle between the line and the plane.

Introducing a parameter $\lambda$, we have the parametric equations:

$x = 2\lambda$, $y = 2\lambda - 6$ and $z = \frac{\lambda + 1}{3}$.

Substituting each of these values into the equation of the plane $3x + y - z = 9$ we obtain:

$$6\lambda + (2\lambda - 6) - \frac{\lambda + 1}{3} = 9$$

i.e. $18\lambda + 6\lambda - 18 - (\lambda + 1) = 27$

$$\therefore \lambda = 2$$

Substituting $\lambda = 2$, we get $x = 4$, $y = -2$ and $z = 1$, i.e. the point of intersection is (4, –2, 1).

Writing the equation of the plane as $\boldsymbol{r} \bullet \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} = 9$

and the equation of the line as $\boldsymbol{r} = \begin{pmatrix} 0 \\ -6 \\ 1/3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 2 \\ 1/3 \end{pmatrix}$,

we have that $\boldsymbol{n} = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}$ and $\boldsymbol{v} = \begin{pmatrix} 2 \\ 2 \\ 1/3 \end{pmatrix}$.

Then $\boldsymbol{v} \bullet \boldsymbol{n} = 6 + 2 - \frac{1}{3} = 7\frac{2}{3}$, $|\boldsymbol{v}| = \sqrt{8\frac{1}{9}}$ and $|\boldsymbol{n}| = \sqrt{11}$.

Hence $\cos\phi = 0.81165....$, $\phi = 35.7°$ and finally $\theta = 54.3°$.

## Exercise C.11.15

1. In each case find:

   i the point of intersection of the line and plane, and

   ii the angle between the line and plane:

| | line | plane |
|---|---|---|
| a | $\boldsymbol{i} + 2\boldsymbol{j} + \lambda(3\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k})$ | $\boldsymbol{r} \bullet (2\boldsymbol{i} + 4\boldsymbol{j} - \boldsymbol{k}) = 28$ |
| b | $\frac{x-1}{2} = y = \frac{3-z}{4}$ | $2x + 3y + z = 11$ |
| c | $\begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix} + \kappa \begin{pmatrix} -1 \\ 3 \\ 3 \end{pmatrix}$ | $\begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$ |
| d | $\frac{x-1}{2} = \frac{y-2}{-3} = \frac{z+3}{4}$ | $2x + 4y - z - 1 = 0$ |

2.

   a A line joins the origin to (6, 10, 8). Find the coordinates of the point where the line cuts the plane $2x + 2y + z = 10$.

   b Find the point where the line joining (2, 1, 3) to (4, –2, 5) cuts the plane $2x + y - z = 3$.

3. Try to describe with words and/or diagrams:

   a the plane $x + y = 6$.

   b the line $x = 4, y = 2z$.

   Now find their point of intersection.

4. Find the distance of the point (–1, –5, –10) from the point of intersection of the line:

   $\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{12}$ and the plane $x - y + z = 5$.

# Intersection of Two Planes

A full treatment of solving simultaneous equations in three unknowns is provided in Chapter A.9. We revisit this area using the development of 3-D geometry that has evolved over this chapter.

If two planes are parallel they will clearly not intersect (unless they coincide), and this case will be identifiable because their respective $\boldsymbol{n}$ vectors will be parallel. For example the planes $2x - y - z = 3$ and $-4x + 2y + 2z = 7$ are parallel because their respective $\boldsymbol{n}$ vectors are $2\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k}$ and $-4\boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k}$ and $-4\boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k} = -2(2\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k})$. If two planes are not parallel they must intersect in a line.

**Example C.11.45**

Find the equation of the line of intersection of the planes

$x+3y+z=5$ (1)

$2x-y-z=1$ (2)

Find also the angle between the two planes.

Our strategy is to eliminate $z$ and hence write $x$ in terms of $y$.

Adding (1) and (2): $3x+2y = 6$ and so $x = \dfrac{6-2y}{3}$.

Now we eliminate $y$ and write $x$ in terms of $z$.

Adding (1) to $3\times$(2): $7x-2z = 8$ and so $x=\dfrac{2z+8}{7}$.

Putting these together into a single equation we have the line

$$x=\frac{6-2y}{3}=\frac{2z+8}{7}$$

Note: having found the line it is worth choosing a simple-valued point on the line, such as (2, 0, 3), and checking that it lies on both planes – which in this case it does.

To find the angle between the planes we find the angle between their normal vectors.

Rewriting the equations as $\boldsymbol{r}\bullet(\boldsymbol{i}+3\boldsymbol{j}+\boldsymbol{k})=5$ and $\boldsymbol{r}\bullet(2\boldsymbol{i}-\boldsymbol{j}-\boldsymbol{k})=1$ we can calculate:

$(\boldsymbol{i}+3\boldsymbol{j}+\boldsymbol{k})\bullet(2\boldsymbol{i}-\boldsymbol{j}-\boldsymbol{k}) = -2$

$|\boldsymbol{i}+3\boldsymbol{j}+\boldsymbol{k}|=\sqrt{11}$

$|(2\boldsymbol{i}-\boldsymbol{j}-\boldsymbol{k})| = \sqrt{6}$

Hence $\cos\theta=\dfrac{-2}{\sqrt{66}}$ and $\theta = 104.3°$.

If the acute angle was required it would be $(180° - 104.3°) = 75.7°$.

## Exercise C.11.16

1. Where possible, find a Cartesian equation of the line of intersection of the two planes and find the acute angle between them:

   a $x+y+z = 3$ and $2x+y+3z = 0$

   b $2x+y+4z = 7$ and $-x+3y+z = -8$

   c $\boldsymbol{r} = \begin{pmatrix}4\\2\\1\end{pmatrix}+p\begin{pmatrix}1\\2\\0\end{pmatrix}+q\begin{pmatrix}-1\\1\\3\end{pmatrix}$ and $\boldsymbol{r} = \begin{pmatrix}0\\2\\0\end{pmatrix}+\lambda\begin{pmatrix}1\\5\\3\end{pmatrix}+\mu\begin{pmatrix}2\\1\\-3\end{pmatrix}$

   d $\boldsymbol{r}\bullet(3\boldsymbol{i}+2\boldsymbol{j}+\boldsymbol{k}) = 10$ and $\boldsymbol{r}\bullet(\boldsymbol{i}-4\boldsymbol{j}-2\boldsymbol{k}) = 8$

2. a Show that the point (5, 2, –1) lies on the line of intersection of the planes $x-3y+z = -2$ and $2x+y+3z = 9$.

   b Show that the line of intersection of the planes $x+y+z=2$ and $2x-y+3z=-4$ is perpendicular to $x = y = z$.

   c Show that the equation of the line of intersection of the planes $4x+4y-5z = 12$ and $8x+12y-13z = 32$ can be written as $\dfrac{x-1}{2} = \dfrac{y-2}{3} = \dfrac{z}{4}$.

3. Find the angle between the lines defined by the intersection of the planes:

$$\begin{cases} x - 2y + z = 0 \\ x + y - z = 0 \end{cases} \text{and} \begin{cases} x + 2y + z = 0 \\ 8x + 12y + 5z = 0 \end{cases}.$$

## Intersection of Three Planes

**Case 1**

When we write the equations of three planes such as:

$$x + y + 2z = 0 \qquad (1)$$

$$2x - y + z = -6 \qquad (2)$$

$$x + y + 3z = 2 \qquad (3)$$

and consider their possible intersection, we are solving a system of equations in three unknowns. There are three possible outcomes:

1. a single solution
2. no solution
3. an infinity of solutions.

Before reading on it is worth playing with three planes (books, pieces of card) and trying to get a clear picture of the geometrical interpretation of each of these possibilities.

If M is the underlying $3 \times 3$ matrix of the system, in our case

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & -1 & 1 \\ 3 & 4 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -6 \\ -6 \end{bmatrix},$$

det M $\neq 0$ leads to outcome (i) and det M $= 0$ leads either to (ii) or to (iii).

$$\det M = 1(-1\times -1 - 4\times 1) - 1(2\times -1 - 3\times 1) + 2(2\times 4 - 3\times -1)$$
$$= 24$$

which means a unique solution, i.e. a single point of intersection.

To find this point we could eliminate $z$ from (1) and (3), then from (2) and (3):

(1) + 2(3) $\qquad 7x + 9y = -12$

(2) + (3) $\qquad 5x + 3y = -12$

and then solve. We get $x = -3$ and $y = 1$, and by going back to (1) we find $z = 1$. Hence the point of intersection is $(-3, 1, 1)$.

(There is considerable freedom as to which variable to eliminate and how to set about eliminating it.)

**Case 2**

Now we look at a case where det M $= 0$ but there is no solution - i.e. the planes have no common point. Such a system is:

$$3x + y + 4z = 8 \qquad (1)$$

$$3x - y - z = 4 \qquad (2)$$

$$x + y + 3z = 2 \qquad (3)$$

We set off in the same way as in Case (1): by eliminating one of the variables in two different ways. For this system the obvious variable to eliminate is $y$:

(1) + (2) $\qquad 6x + 3z = 12$

(2) + (3) $\qquad 4x + 2z = 6$

The first equation is equivalent to $2x + z = 4$ and the second is equivalent to $2x + z = 3$. The equations are inconsistent with each other and there is no solution to the system. The three dimensional picture is of three planes that have no point of intersection.

**Case 3**

In this system check that det M $= 0$:

$$3x - y - z = 1 \qquad (1)$$

$$x + 2y + z = 4 \qquad (2)$$

$$x - 5y - 3z = -7 \qquad (3)$$

We could eliminate $x$ in two ways:

$3 \times (2) - (1) \qquad 7y + 4z = 11$

(2) - (3) $\qquad 7y + 4z = 11.$

It is important to be clear what this means: if we choose *any* $y$ and $z$ satisfying $7y + 4z = 11$ we can find the value of $x$ such that all three equations (1, 2 and 3) are satisfied. An example would be $y = z = 1$, leading to $x = 1$; check that all three equations are satisfied. But if we chose to satisfy $7y + 4z = 11$ with $y = 5$, $z = -6$ we get $x = 0$, and again all three equations are satisfied.

Clearly we could find as many solutions as we wanted.

Solution is $\left(\frac{\lambda+6}{7}, \frac{11-4\lambda}{7}, \lambda\right)$.

To summarise: if det M = 0 there are two possibilities.

a When we eliminate one of the variables in two different ways and we get two inconsistent equations in the other two variables, then we have no solution. The three dimensional picture of this is three planes that fail to intersect.

b When we eliminate one of the variables in two different ways and we get two identical equations in the other two variables, then we have an infinity of solutions. The three dimensional picture of this is three planes intersecting in a line. (To find the equation of the line, find the equation of the line of intersection of any two of the planes.)

## Exercise C.11.17

1. Three planes can fail to have any point of intersection if two or more of them are parallel.

   Describe a situation where three planes fail to intersect but no pair of planes is parallel.

2. Analyse Case 2 in a little more detail:

   a Find a Cartesian equation of the line of intersection of $|u| = 0.5, |v| = 12$ and $3x - y - z = 4$.

   b Show that this line is parallel to $x + y + 3z = 2$.

3. Analyse Case 3 in a little more detail:

   a Find a Cartesian equation of the line of intersection of $3x - y - z = 1$ – (1) and $x + 2y + z = 4$ – (2).

   b Show that this line lies in the plane $x - 5y - 3z = -7$ – (3).

   c Show that (1) = 2 × (2) + (3).

4. Classify each set of planes as:

   i intersecting in a single point, in which case give its coordinates, or

   ii no point of intersection, or

   iii intersecting in a line, in which case give a Cartesian equation.

   a
   $$\begin{aligned} x + y - z &= 10 \\ 2x - 3y + z &= 5 \\ x - 4y + 2z &= 6 \end{aligned}$$

   b
   $$\begin{aligned} x + y + z &= 10 \\ 2x - y &= 9 \\ -x + 3y + 4z &= 14 \end{aligned}$$

   c
   $$\begin{aligned} x + 2y - z &= 10 \\ 3x - y + z &= 11 \\ 2x + y + 4z &= -1 \end{aligned}$$

   d
   $$\begin{aligned} 2x + y + 3z &= -5 \\ x - 2y + 2z &= -9 \\ 3x + 4y + 4z &= -1 \end{aligned}$$

5. *This question involves concepts from the whole of this chapter.*

   OBCDEFGH is a cuboid with O(0, 0, 0); B(0, 0, 3); C(4, 0, 3); D(4, 0, 0); E(4, 2, 0); F(0, 2, 0); G(0, 2, 3); H(4, 2, 3).

   a Sketch the cuboid.

   b Find parametric forms for the equations of lines OH and BE. Show that the two lines intersect at the point (2, 1, 1.5).

   c Find the Cartesian equation of plane FHD. (A parametric form is $\boldsymbol{r} = \mathbf{OF} + s\mathbf{FH} + t\mathbf{FD}$. Now convert to Cartesian form.)

   d Find the coordinates of the point of intersection of line BE and plane FHD, and also the angle between the line and plane.

   e Find the angle between plane FHD and plane GHCB.

6. Show that the equations:
   $$\begin{aligned} x + y - z &= -1 \\ 5x + 3y + z &= 3 \\ 2x + y + z &= a \end{aligned}$$
   are inconsistent for $a = 1$ and describe this situation geometrically in terms of intersecting planes.

7. Find the value of $k$ for which the system of equations:
   $$\begin{aligned} 8x + 3y + z &= 12 \\ x + 2z &= 3 \\ 2x + y - z &= k \end{aligned}$$

represents three planes that intersect in a common line and find the vector equation in parametric form of the line of intersection.

8. The planes $x-3y-z=0$ and $3x-5y-z=0$ intersect in a line, L, that passes through the origin.

a Find the vector product of the normals to both planes.

b Hence, find the vector equation of L.

c Find the value of $k$ for which the system of equations:

$$\begin{aligned} x-3y-z &= 0 \\ 3x-5y-z &= 0 \\ -x+ky+2z &= k^2-4 \end{aligned}$$

has:

i no real solutions.

ii infinitely many solutions.

iii a unique solution.

9. a On a set of axes, sketch the planes $x+y=2a$, $y+z=2b$, $z+x=2c$.

b Find where the planes meet, i.e. solve the system of equations:

$$\begin{aligned} x+y &= 2a \\ y+z &= 2b \\ z+x &= 2c \end{aligned}$$

c Hence, deduce the solution to the system:

$$x+y=\frac{2}{a},\; y+z=\frac{2}{b},\; z+x=\frac{2}{c}$$

10. a Find the two values of $k$ for which the planes with equations $-x+y+2z=3$, $kx+y-z=3k$ and $x+3y+kz=13$ have no unique solution.

b Show that for one value of $k$, there are in fact no solutions.

c Show that for the other value of $k$, the planes meet along a line. Find the Cartesian equation of this line.

11. Show that the equation for the plane passing through the point $M(x_0, y_0, z_0)$ and perpendicular to the planes $a_1x+b_1y+c_1z=d_1$ and $a_2x+b_2y+c_2z=d_2$ can be written in the form:

$$\begin{vmatrix} x-x_0 & y-y_0 & z-z_0 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0.$$

12. Show that the equation for the plane passing through the points $M(x_0, y_0, z_0)$, $N(x_1, y_1, z_1)$ and perpendicular to the plane $ax+by+cz=d$ can be written in the form:

$$\begin{vmatrix} x-x_0 & y-y_0 & z-z_0 \\ x_1-x_0 & y_1-y_0 & z_1-z_0 \\ a & b & c \end{vmatrix} = 0.$$

13. Show that the equation for the plane passing through the point $M(x_0, y_0, z_0)$ and parallel to the straight lines:

$$\text{L}_1: \boldsymbol{r} = \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} + \lambda \begin{pmatrix} l_1 \\ m_1 \\ n_1 \end{pmatrix} \text{ and } \text{L}_2: \boldsymbol{r} = \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} + t \begin{pmatrix} l_2 \\ m_2 \\ n_2 \end{pmatrix}$$

may be written in the form $\begin{vmatrix} x-x_0 & y-y_0 & z-z_0 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$.

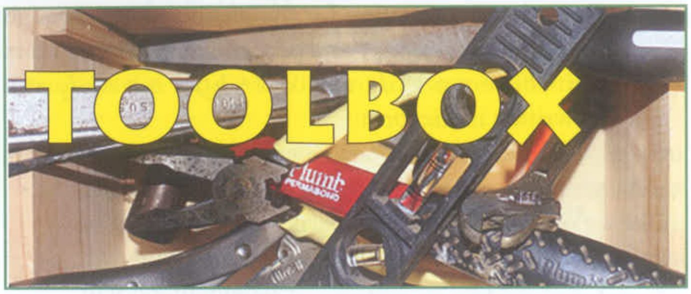
14. Show that the equation for the plane which contains the lines

$$\text{L}_1: \boldsymbol{r} = \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} + \lambda \begin{pmatrix} l \\ m \\ n \end{pmatrix} \text{ and } \text{L}_2: \boldsymbol{r} = \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} + t \begin{pmatrix} l \\ m \\ n \end{pmatrix}$$

may be written in the form

$$\begin{vmatrix} x-a_1 & y-b_1 & z-c_1 \\ a_2-a_1 & b_2-b_1 & c_2-c_1 \\ l & m & n \end{vmatrix} = 0.$$

Answers




## Applications

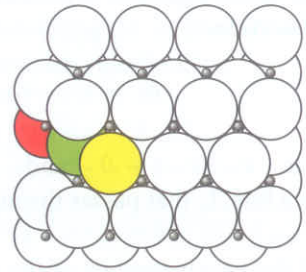
### Crystals

The beautifully regular shapes of crystals arise naturally when molten minerals solidify or when solutions are concentrated by evaporation.

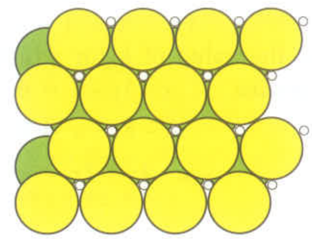
The regular shapes occur when the atoms (ions, molecules) 'close-pack' to form arrangements like a stack of tennis balls in a sports shop.

The techniques discussed in this section should enable you to investigate the shapes that arise when identical spheres form such crystals.

There are two types of stack:



**ABC**
3rd layer directly above
the hollows ○ in the 1st layer **ABC**

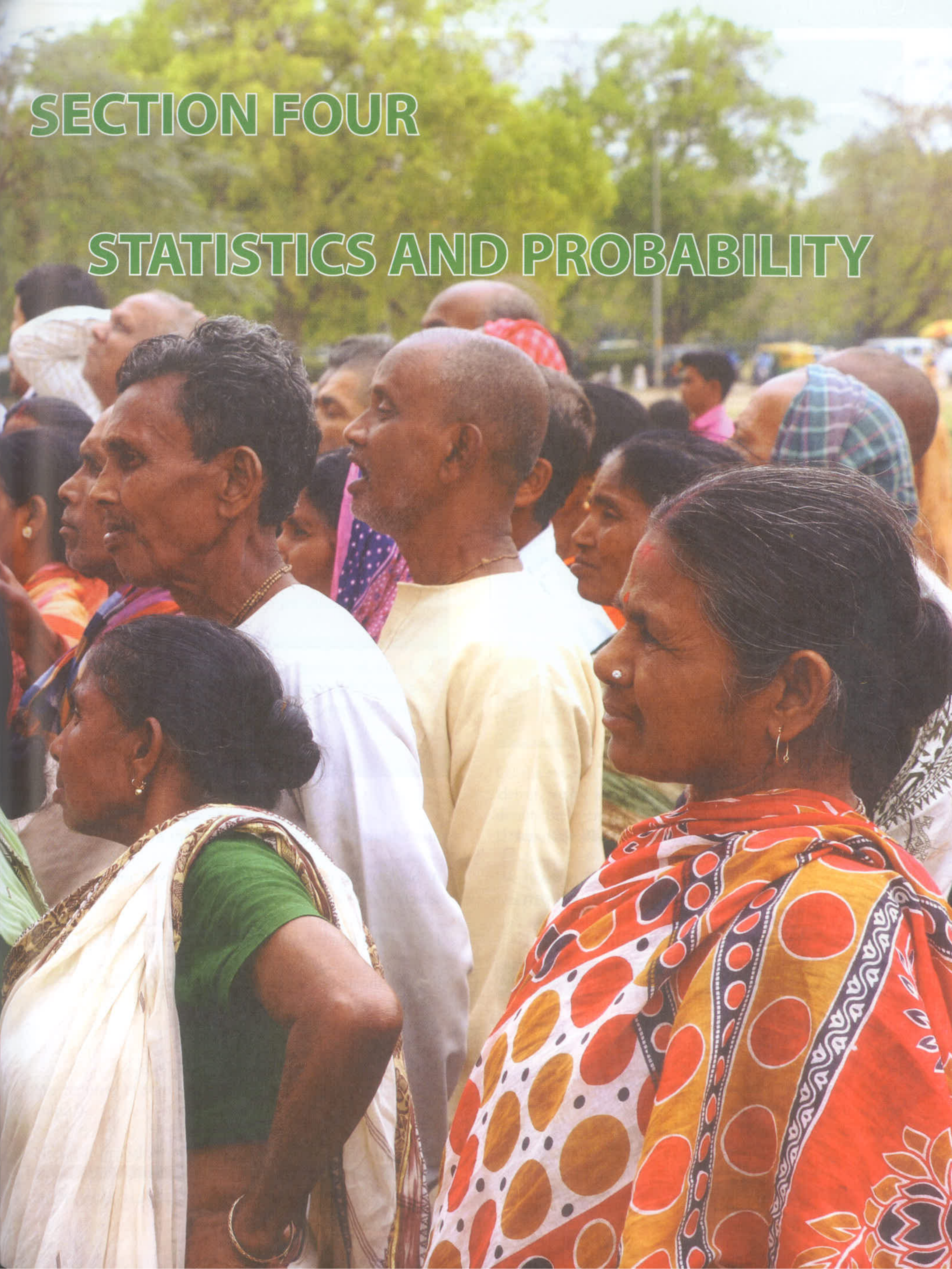


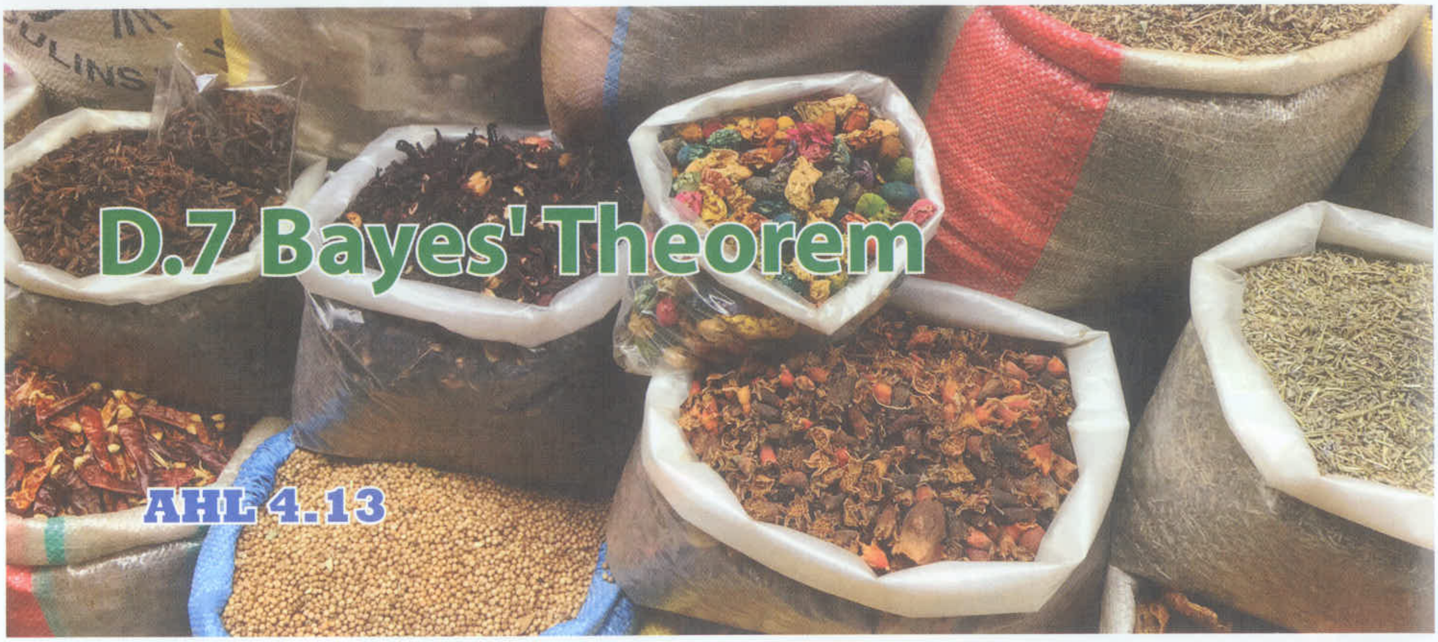
**ABAB**
3rd layer directly above
the hollows ● in the 2nd layer **ABA**

What two crystal forms result from these two arrangements?

# SECTION FOUR

# STATISTICS AND PROBABILITY





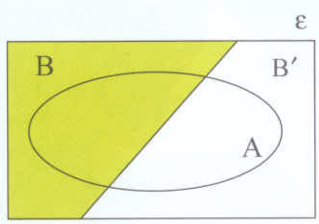
# D.7 Bayes' Theorem

AHL 4.13

## Bayes' Theorem

### Law of total probability

Using the Venn diagram, for any event A, we have that




$A = A \cap \varepsilon = A \cap (B \cup B')$

$= (A \cap B) \cup (A \cap B')$

As these two events are mutually exclusive, we have:

$$P(A) = P(A \cap B) + P(A \cap B')$$

However,

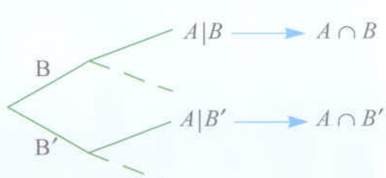
$P(A|B) = \dfrac{P(A \cap B)}{P(B)} \Rightarrow P(A \cap B) = P(B) \times P(A|B)$ and

$P(A|B') = \dfrac{P(A \cap B')}{P(B')} \Rightarrow P(A \cap B') = P(B') \times P(A|B')$,

which leads to the **Law of Total Probability**.

$$P(A) = P(B) \times P(A|B) + P(B') \times P(A|B')$$

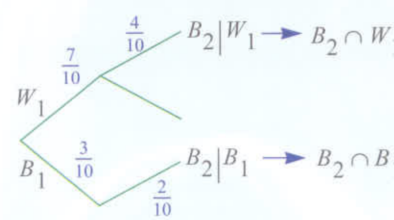
Although this expression may look daunting, in fact, it represents the result that we would obtain if a tree diagram was used.




**Example D.7.1**

A box contains 3 black cubes and 7 white cubes. A cube is drawn from the box. Its colour is noted and a cube of the other colour is then added to the box. A second cube is then drawn. What is the probability that the second cube selected is black?

We begin by setting up a tree diagram, where $B_i$ denotes the event "A Black cube is observed on *i*th selection" and $W_i$ denotes the event "A White cube is observed on *i*th selection".




A black cube could have been observed on the second selection if:

**i** the first cube selected was white (i.e. $B_2|W_1$), or

**ii** the first cube selected was black (i.e. $B_2|B_1$).

Therefore, $P(B_2) = P(B_2 \cap W_1) + P(B_2 \cap B_1)$

$= \dfrac{7}{10} \times \dfrac{4}{10} + \dfrac{3}{10} \times \dfrac{2}{10}$

$= \dfrac{17}{50}$

## Bayes' Theorem for two events

As we saw earlier, conditional probability provides a means by which we can adjust the probability of an event in light of new information. Bayes' Theorem, developed by Rev. Thomas Bayes, pictured, (1702–1761), does the same thing, except this time it provides a means of adjusting a set of associated probabilities in the light of new information.

For two events, we have:

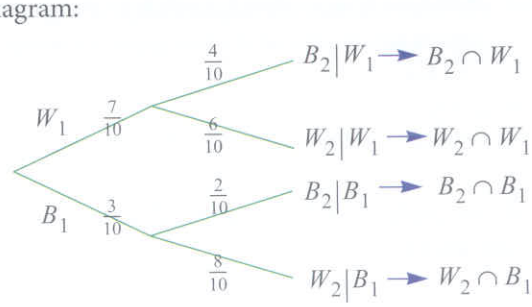
$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A) \times P(B|A)}{P(A) \times P(B|A) + P(A') \times P(B|A')}$$

Again, the formula may seem daunting, however, it is only making use of a tree diagram.

### Example D.7.2

A box contains 3 black cubes and 7 white cubes. A cube is drawn from the box. Its colour is noted and a cube of the other colour is then added to the box. A second cube is then drawn. If both cubes are of the same colour, what is the probability that both cubes were in fact white?

Following on from the previous example, we have the same tree diagram:




We require: P(Both white **given** that both are of the same colour)

Now, the probability that they are of the same colour is given by the probability that they are **both white or both black**, i.e. $P((W_2 \cap W_1) \cup (B_2 \cap B_1))$.

Next: P(Both White **given** both are the same colour)

$$= P(W_2 \cap W_1 | (W_2 \cap W_1) \cup (B_2 \cap B_1))$$

$$= \frac{P((W_2 \cap W_1) \cap ((W_2 \cap W_1) \cup (B_2 \cap B_1)))}{P((W_2 \cap W_1) \cup (B_2 \cap B_1))}$$

$$= \frac{P(W_2 \cap W_1)}{P(W_2 \cap W_1) + P(B_2 \cap B_1)}$$

$$= \frac{P(W_2|W_1) \times P(W_1)}{P(W_2|W_1)P(W_1) + P(B_2|B_1)P(B_1)}$$

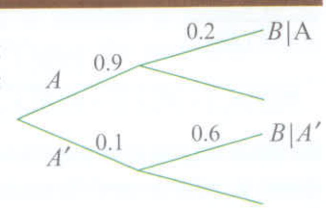
$$= \frac{\frac{6}{10} \times \frac{7}{10}}{\frac{6}{10} \times \frac{7}{10} + \frac{2}{10} \times \frac{3}{10}}$$

$$= \frac{7}{8}$$

### Example D.7.3

In a small country town, it was found that 90% of the drivers would always wear their seatbelts. On 60% of occasions, if a driver was not wearing a seatbelt they would be fined for speeding. If they were wearing a seatbelt, they would be fined for speeding 20% of the time. Find the probability that a driver who was fined for speeding was wearing a seatbelt.

Let the event $A$ denote the event 'driver wears a seatbelt' and $B$ denote the event 'Driver speeds'. Using a tree diagram we have:




We need to find, P(Driver was wearing a seatbelt | driver was booked for speeding):

$$= P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$= \frac{P(A) \times P(B_|A)}{P(A) \times P(B|A) + P(A') \times P(B|A')}$$

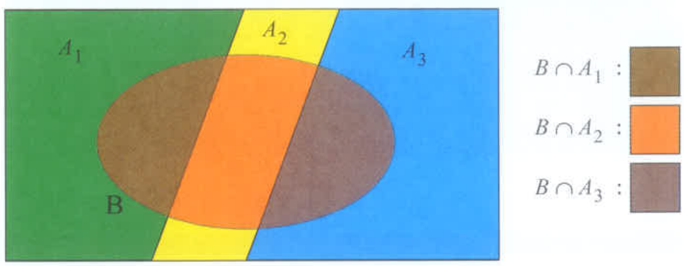
$$= \frac{0.9 \times 0.2}{0.9 \times 0.2 + 0.1 \times 0.6} = \frac{18}{24}$$

So, P(that a driver who was booked for speeding was in fact wearing a seatbelt) = 0.75

## Bayes' Theorem for three events

So far we have used Bayes' Theorem for the case when the sample space is partitioned in two events, A and $A'$, where $A \cup A' = U$. However, this can be easily extended to the situation when the sample may be partitioned into many events. That is, $A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n = U$ where each of the events $A_i$ are mutually exclusive.

So, let's consider the case when there are 3 events, so that the event A can be partitioned into three exhaustive, mutually exclusive subsets, i.e. $B = (B \cap A_1) \cup (B \cap A_2) \cup (B \cap A_3)$.



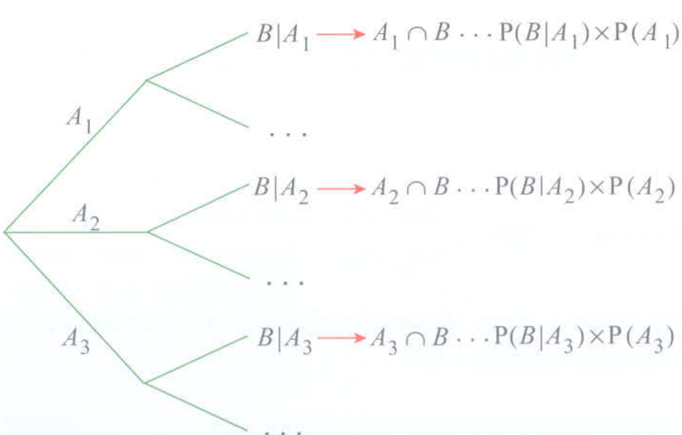

Then,

$$\begin{aligned} P(B) &= P((B \cap A_1) \cup (B \cap A_2) \cup (B \cap A_3)) \\ &= P(B \cap A_1) + P(B \cap A_2) + P(B \cap A_3) \\ &= \frac{P(B \cap A_1)}{P(A_1)} \times P(A_1) + \frac{P(B \cap A_2)}{P(A_2)} \times P(A_2) + \ldots \\ &\qquad \ldots + \frac{P(B \cap A_3)}{P(A_3)} \times P(A_3) \\ &= P(B|A_1) \times P(A_1) + P(B|A_2) \times P(A_2) + P(B|A_3) \times P(A_3) \end{aligned}$$

Therefore, we have that:

$$P(A_1|B) = \frac{P(A_1 \cap B)}{P(B)} = \frac{P(B|A_1) \times P(A_1)}{P(B|A_1) \times P(A_1) + P(B|A_2) \times P(A_2) + P(B|A_3) \times P(A_3)}$$

As daunting as this expression may appear, all that we have done is add a new branch to our existing tree diagram. Everything remains the same. Making use of a tree diagram to help us evaluate the required probabilities is always useful. Such a diagram would have the following structure:




### Example D.7.4

It is found that the population within a particular state has a certain blood disease. Of this population, 2% have a serious form of the disease; 5% have a mild form; while 93% do not have it at all.

In carrying new trials, a new blood test is used, recording a positive result 92% of the time if the patient has the serious form of the disease; recording a positive result 60% of the time if the patient has the mild form of the disease, and recording a positive result 10% of the time if the patient does not have the disease.

A patient has just tested positive. What is the probability that this patient has the serious form of the disease?

Using the notation just discussed, we let $A_1$ denote the event 'Patient has disease in serious form', $A_2$ denote the event 'Patient has disease in mild form' and $A_3$ denote the event 'Patient does not have the disease'. Let B denote the event 'Records positive blood test'.

This gives, $P(A_1) = 0.02$, $P(A_2) = 0.05$ and $P(A_3) = 0.93$

$P(B|A_1) = 0.92$, $P(B|A_2) = 0.60$, $P(B|A_3) = 0.10$

Then, as:

$$\begin{aligned} P(B) &= P(B|A_1) \times P(A_1) + P(B|A_2) \times P(A_2) + P(B|A_3) \times P(A_3) \\ &= 0.92 \times 0.02 + 0.6 \times 0.05 + 0.10 \times 0.93 \\ &= 0.1414 \end{aligned}$$

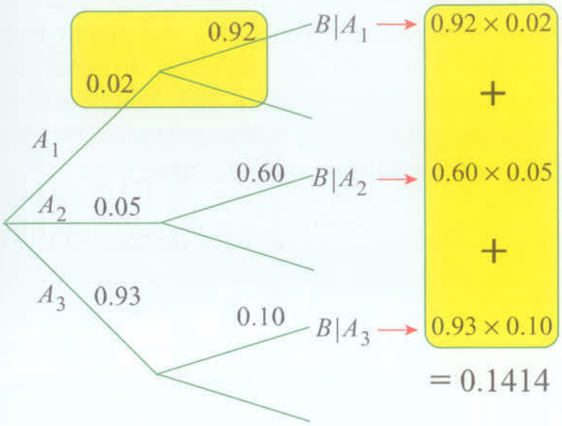
Using Bayes' Theorem, we have:

$$\begin{aligned} P(A_1|B) &= \frac{P(A_1 \cap B)}{P(B)} \\ &= \frac{P(B|A_1)P(A_1)}{P(B|A_1) \times P(A_1) + P(B|A_2) \times P(A_2) + P(B|A_3) \times P(A_3)} \\ &= \frac{0.92 \times 0.02}{0.1414} \\ &= 0.1301 \end{aligned}$$

Of course, we could have drawn a tree diagram to help with the example above:

As in the case for two events, drawing a tree diagram works very well. However, one needs to make sure to allocate the correct probabilities to the appropriate branches.

Next we consider a problem with three consecutive events producing three levels of branches, each identifying two possible outcomes.




### Example D.7.5

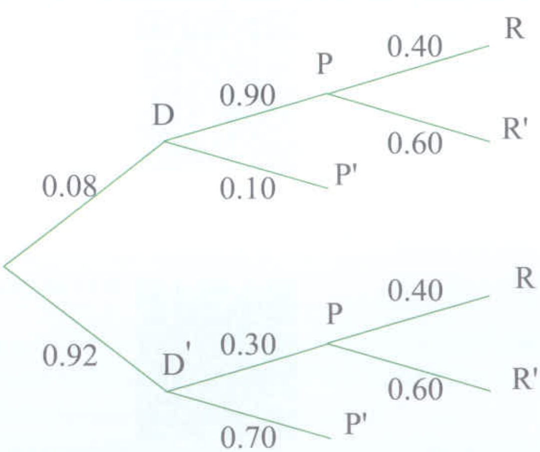
It is found that 8% of the population in a particular city has a disease, 'D'. A test is administered with the following outcomes: test returns a positive result 90% of the time if a person has the disease and returns a positive result 30% of the time if the person does not have the disease.

If a positive result is recorded a second drug is administered, which will cure the disease. However, this drug has the side effect of producing a nasty rash 40% of the time.

A randomly selected person is found to have the rash. What is the probability s/he had the disease to start with?

Let 'D' denote the event that a person has the disease, 'P' denote the event that a person will produce a positive reading and 'R' the event that the person develops a rash.

From the given information, we can produce the following tree diagram (leaving out irrelevant information):




$$P(D|R) = \frac{P(D \cap R)}{P(R)}$$

$$= \frac{0.08 \times 0.90 \times 0.40}{0.08 \times 0.90 \times 0.40 + 0.92 \times 0.30 \times 0.40}$$

$$= 0.2069$$

Again, notice how a tree diagram was most helpful in producing a neat and compact solution.

## Exercise D.7.1

1. Machine A produces 40% of the daily output of a factory but 3% of the items manufactured from this machine are defective. Machine B produces 60% of the daily output of the same factory but 5% of the items manufactured from this machine are defective.

   a An item is selected at random. Find the probability that it is defective.

   b An item is selected and is found to be defective. Find the probability that it came from machine B.

2. At the Heights International School, it is found that 12% of the male students and 7% of the female students are taller than 1.8 m. Sixty per cent of the school is made up of female students.

   a A student selected at random is found to be taller than 1.8m. What is the probability that the student is a female?

   b A second student selected at random is found to be shorter than 1.8m. What is the probability that the student is a male?

3. A box contains 4 black cubes and 6 white cubes. A cube is drawn from the box. Its colour is noted and a cube of the other colour is then added to the box. A second cube is then drawn.

   a If both cubes are of the same colour, what is the probability that both cubes were in fact white?

   b The first cube is replaced before the second cube is added to the box. What is the probability that both cubes were white given that both cubes were of the same colour?

4. An urn, labelled A, contains 8 cards numbered 1 through 8 whilst a second urn, labelled B, contains five cards numbered 1 through five. An urn is selected at random and from that urn a card is selected. Find the probability that the card came from urn A given that it is an even numbered card.

5. An event $A$ can occur only if one of the mutually exclusive events $B_1$, $B_2$ or $B_3$ occurs. Show that
$P(A) =$
$P(B_1) \times P(A|B_1) + P(B_2) \times P(A|B_2) + P(B_3) \times P(A|B_3)$

b Of the daily output, machines A and B produce items of which 2% are defective, whilst machine C produces items of which 4% are defective. Machines B and C produce the same number of items, whilst machine A produces twice as many items as machine B.

i An item is selected at random. Find the probability that it is defective.

ii An item is selected and is found to be defective. Find the probability that it came from machine B.

6. A box contains $N$ coins, of which $m$ are fair coins whilst the rest are double-headed coins.

a A coin is selected at random and tossed.

i What is the probability of observing a head?

ii Given that a head was observed, what is the probability that a double–headed coin was selected?

b This time, a coin is selected at random and tossed $n$ times. What is the probability that it is a fair coin, if it shows up heads on all $n$ tosses?

7. A population of mice is made up of 75% that are classified as 'M+', of which, 30% have a condition classified as 'N–'. Otherwise, all other mice have the 'N–' condition. A mouse selected at random is classified as having the 'N–' condition. What is the probability that the mouse comes from the 'M+' classification group?

8. A survey of the adults in a town shows that 8% have liver problems. Of these, it is also found that 30% are heavy drinkers, 60% are social drinkers and 10% are non-drinkers. Of those that did not suffer from liver problems, 5% are heavy drinkers, 65% are social drinkers and 30% do not drink at all.

a An adult is selected at random. What is the probability that this person is a heavy drinker?

b If a person is found to be a heavy drinker, what is the probability that this person has liver problems?

c If a person is found to have liver problems, what is the probability that this person is a heavy drinker?

d If a person is found to be a non-drinker, what is the probability that this person has liver problems?

9. The probability that a person has a deadly virus is 5 in one thousand. A test will correctly diagnose this disease 95% of the time and incorrectly on 20% of occasions.

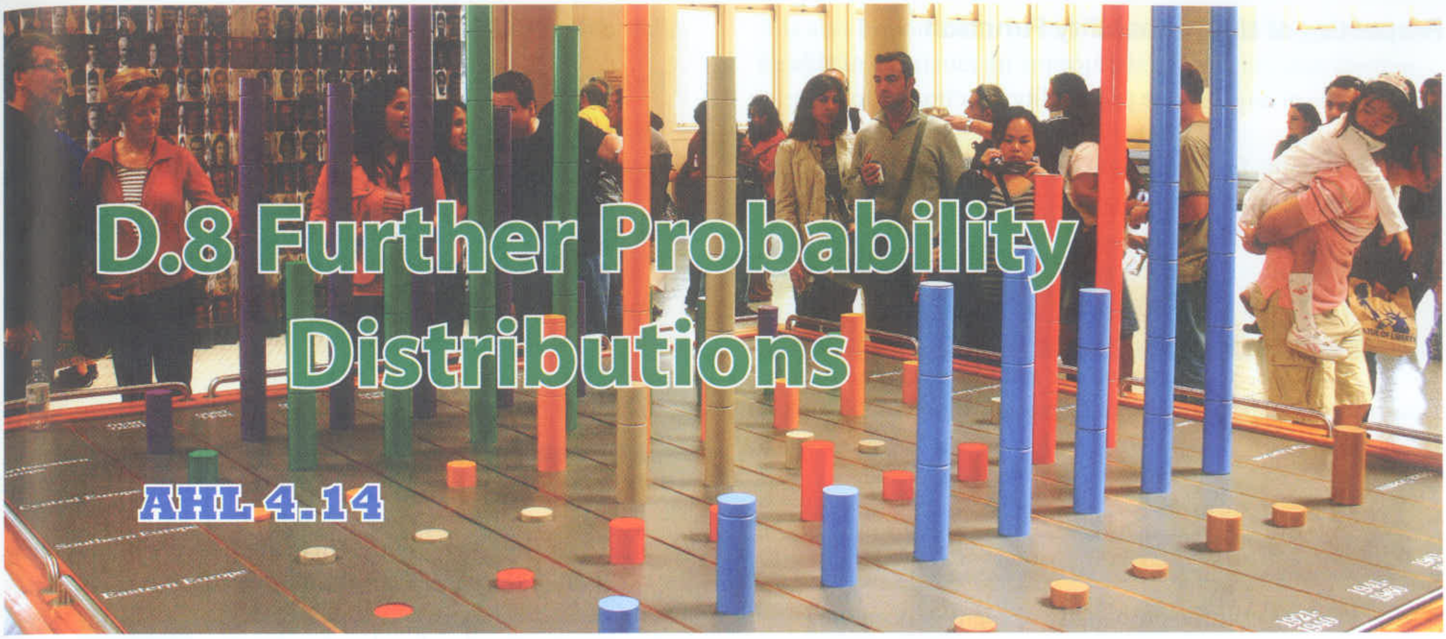
a Find the probability of this test giving a correct diagnosis.

b Given that the test diagnoses the patient as having the disease, what is the probability that the patient does not have the disease?

c Given that the test diagnoses the patient as not having the disease, what is the probability that the patient does have the disease?

Extra questions

Answers



# D.8 Further Probability Distributions

AHL 4.14

We have already encountered the idea of a probability distribution - the binomial distribution, the normal distribution etc. Also, we have noted that there are two main types of distribution: those that deal with discrete data and those that use continuous data.

In this chapter, we will look at distributions that do not fall within these major classifications.

## Discrete Distributions

We can describe a discrete random variable by making use of its probability distribution. That is, by showing the values of the random variable and the probabilities associated with each of its values.

A probability distribution can be displayed in any one of the following formats:

1. Tabular form

2. Graphical representation(With the probability value on the vertical axis, and the values of the random variable on the horizontal axis.)

3. Function (A formula that can be used to determine the probability values.)

### Example D.8.1

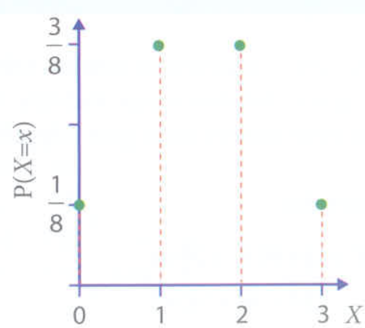
Use each of the probability distribution representations discussed to display the results of the experiment where a coin is tossed three times in succession.

Let the random variable X denote the number of heads observed in three tosses of a coin.

1. Tabular form:

| $X$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $P(X=x)$ | $\frac{1}{8}$ | $\frac{3}{8}$ | $\frac{3}{8}$ | $\frac{1}{8}$ |

2. Graphical representation




3. Function

$$P(X=x)=\binom{3}{x}\left(\frac{1}{2}\right)^{x}, x=0,1,2,3$$

where $\binom{3}{x}=\frac{3!}{(3-x)!x!}$

## Properties of the Probability Function

The features of any discrete probability function as follows:

1. The probability for any value of $X$ must always lie between 0 and 1 (inclusive).

That is, $0 \le P(X = x_i) \le 1$ for all values of .

2. For the $n$ mutually exclusive and exhaustive events, $A_1, A_2, \ldots A_n$ that make up the sample space, the sum of the corresponding probabilities must be 1.

That is:

$$P(X = x_i) = P(X = x_1) + P(X = x_2) + \ldots + P(X = x_n) = 1$$

Where $P(X = x_i)$ is the probability of event $A_i$ occurring.

Any function that does not obey these two rules cannot be a probability function.

**Example D.8.2**

Consider the random variable $X$ with probability function defined by:

$P(X = 0) = 2k, P(X = 1) = 3k$, and $P(X = 2) = 5k$

Determine the value of $k$.

Because we are given that this is a probability function, then summing all the probabilities must give a result of 1.

Therefore we have that:

$$\begin{aligned} P(X = 0) + P(X = 1) + P(X = 2) &= 1 \\ 2k + 3k + 5k &= 1 \\ 10k &= 1 \\ k &= 0.1 \end{aligned}$$

**Example D.8.3**

The probability distribution of the random variable $X$ is represented by the function:

$$P(X = x) = \frac{k}{x^2}, x = 1, 2, 3$$

Find:

a $k$ b $P(2 \le X \le 3)$

a Using the fact that the sum of all the probabilities must be 1, we have:

$$\begin{aligned} P(X = 1) + P(X = 2) + P(X = 3) &= 1 \\ \frac{k}{1} + \frac{k}{4} + \frac{k}{9} &= 1 \\ \frac{49k}{36} &= 1 \\ k &= \frac{36}{49} \end{aligned}$$

b
$$\begin{aligned} P(2 \le X \le 3) &= P(X = 2) + P(X = 3) \\ &= \frac{k}{4} + \frac{k}{9} \\ &= \frac{13k}{36} \\ &= \frac{13}{49} \end{aligned}$$

**Example D.8.4**

A discrete random variable $X$ has a probability distribution defined by the function:

$$P(X = x) = \begin{pmatrix} 4 \\ x \end{pmatrix} \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{4-x}, x = 0, 1, 2, 3, 4$$

a Display this distribution using:

i a table form ii a graphical form.

b Find:

i $P(X = 2)$ ii $P(1 \le X \le 3)$.

a i We begin by evaluating the probability for each value of $x$:

$$P(X=0)=\binom{4}{0}\left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right)^{4-0}=1\times1\times\frac{16}{81}=\frac{16}{81}$$

$$P(X=1)=\binom{4}{1}\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^{4-1}=4\times\frac{1}{3}\times\frac{8}{27}=\frac{32}{81}$$

$$P(X=2)=\binom{4}{2}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^{4-2}=6\times\frac{1}{9}\times\frac{4}{9}=\frac{24}{81}$$

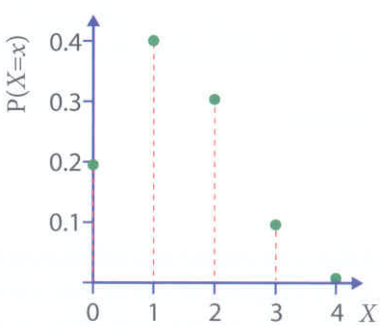
$$P(X=3)=\binom{4}{3}\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^{4-3}=4\times\frac{1}{27}\times\frac{2}{3}=\frac{8}{81}$$

$$P(X=4)=\binom{4}{4}\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)^{4-4}=1\times\frac{1}{81}\times1=\frac{1}{81}$$

We can now set up this information in a table:

| $X$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $P(X=x)$ | $\frac{16}{81}$ | $\frac{32}{81}$ | $\frac{24}{81}$ | $\frac{8}{81}$ | $\frac{1}{81}$ |

ii Using the table found in part i, we can construct the following graph:




b i From the probability table, we have that:

$$P(X=2)=\frac{24}{81}$$

ii The statement $P(1\le X\le 3)$ requires that we find the probability of the random variable $X$ taking on the values 1, 2 or 3. This amounts to evaluating the sum of the corresponding probabilities.

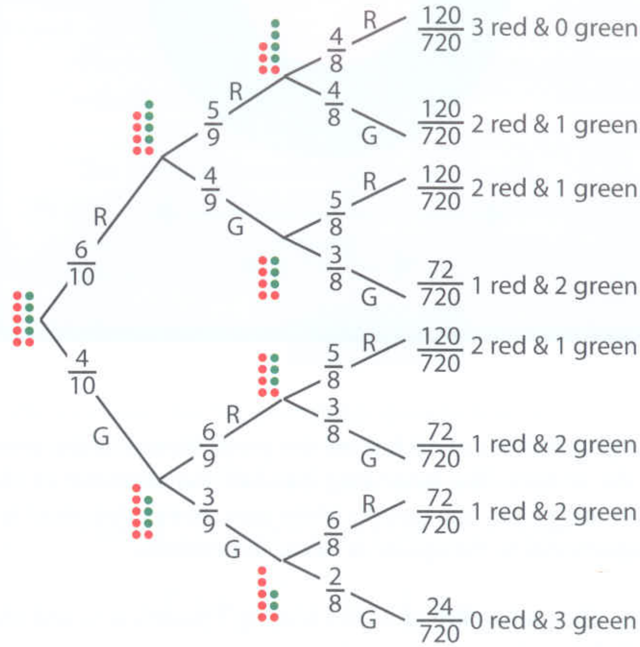
$$\begin{aligned}P(1\le X\le3)&=P(X=1)+P(X=2)+P(X=3)\\&=\frac{32}{81}+\frac{24}{81}+\frac{8}{81}\\&=\frac{64}{81}\end{aligned}$$

It is also important to be able to take a physical situation and be able to construct an appropriate probability distribution.

**Example D.8.5**

A bag contains 6 red balls and 4 green balls. Three balls are drawn without replacement. Draw up a table for the distribution of the number of red balls in the sample.

A tree diagram should help track the sample space for each draw:




Defining X as the number of red balls in the sample, we have:

$$P(X=0)=\frac{24}{720}=\frac{1}{30}$$

$$P(X=1)=\frac{72}{720}+\frac{72}{720}+\frac{72}{720}=\frac{3}{10}$$

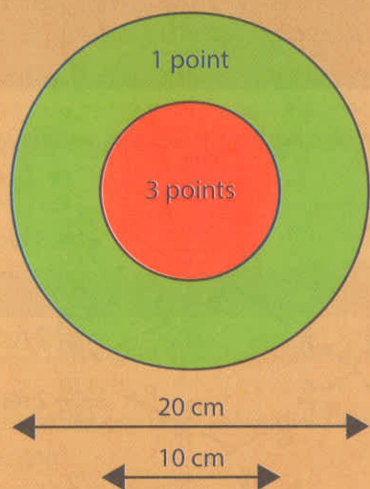
$$P(X=2)=\frac{120}{720}+\frac{120}{720}+\frac{120}{720}=\frac{1}{2}$$

$$P(X=3)=\frac{120}{720}=\frac{1}{6}$$

Note that the probabilities add to 1.

### Example D.8.6

Two darts are thrown at the target shown. Both hit the target, but land at random. The points for each dart are added. Tabulate the probability distribution of possible scores.




The probabilities of each score are proportional to the areas of the sectors. The inner ring has half the diameter of the outer and so has one quarter of the area. Recall that areas are proportional to the square of linear dimensions.

Thus, the probability of a dart scoring 3 points is ¼ and the probability of scoring 1 point is ¾.

P(2 inners, 6 points) = $(\frac{1}{4})^2 = \frac{1}{16}$.

P(an inner, then an outer, 4 points) = $\frac{1}{4} \times \frac{3}{4} = \frac{3}{16}$.

P(an outer, then an inner, 4 points) = $\frac{3}{4} \times \frac{1}{4} = \frac{3}{16}$.

P(2 inners, 2 points) = $(\frac{3}{4})^2 = \frac{9}{16}$.

Thus the distribution is:

$P(X = 2) = \frac{9}{16}$

$P(X = 4) = \frac{3}{16} + \frac{3}{16} = \frac{6}{16} = \frac{3}{8}$

$P(X = 6) = \frac{1}{16}$

## Exercise D.8.1

1. $P(X = x) = \frac{k}{x}, x = 1,2,3,4$ is a probability distribution.

   Find:

   a $k$.

   b $P(X = 2)$.

   c $P(1 \leq X \leq 3)$.

2. $P(X = x), x = 1,2,3,4$ is a probability distribution such that:

   $$P(X=1) = P(X=2) = 2 \times P(X=3) = 2 \times P(X=3)$$

   Find $P(X = 3)$.

3. Two fair six sided dice are rolled and the numbers uppermost are added. If X is the random variable defined by this total:

   a Tabulate the probability distribution of X.

   b Find $P(X \geq 10)$

4. The probability that a jet engine will fail during a flight is 1 in 60 000. A four engined jet can fly safely with two functioning engines. What is the probability of a crash resulting from engine failure?

5. A probability distribution is defined by:

   $P(X = x) = \frac{k}{x!}, x = 0,1,2,3$. Find:

   a $k$.

   b $P(X = 2)$.

6. A box contains 5 balls labelled '2', 3 balls labelled '3' and 2 balls labelled '4'. A ball is drawn at random, the number is noted and the ball is replaced. A second ball is drawn and its number is added to the first.

a List the possible total scores.

b Tabulate the probability distribution of these scores.

7. A probability distribution is defined by:

$$P(X=x)=\frac{k}{(x+1)^2}, x=0,1,2,3$$

Find the value of $k$.

8. The number of meteorites observed during a 24-hour period is given by the random variable $X$, having a probability distribution:

$$P(X=x)=\frac{2^x}{x!}e^{-2}, x=0,1,2,3...$$

Find the probability that more than 3 meteorites are observed in a 24-hour period.

9. Prove that if $0 \le p \le 1$, then:

$$P(X=x)=\binom{n}{x}p^x(1-p)^{n-x}, x=0,1,2,3...,n$$

defines a probability distribution.

10. Prove that if $0 \le p \le 1$, then:

$$P(X=x)=p^{x-1}(1-p), x=1,2,3...$$

defines a probability distribution.

11. $X$ is a the random variable defined as the number of rolls of two fair six sided dice before a 2 and a 1 is rolled for the first time.

a Define the distribution of $X$.

b Find $P(X>2)$.

## Continuous Random Variables

The concept of a continuous variable was introduced in Chapter D6 of the SL book when we met the idea that probability can be measured by finding the area under a curve – the normal curve. In this section, we will look at examples in which these areas are found by integration. Some of the examples may best be tackled after studying the remaining chapters on calculus.

A continuous probability density function (or continuous probability distribution), $f(x)$, is a function satisfying the following properties:

1. The variable is continuous and can assume all real-valued numbers.

2. The function is non-negative, i.e. $f(x) \ge 0$.

3. The total area contained between the graph and the horizontal axis is 1.

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

Often we have that the probability density function is defined over some interval, $a \le x \le b$, so that $f(x) \ge 0$ for $a \le x \le b$ and $f(x) = 0$ elsewhere. This means that is rewritten as:

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{a} f(x)dx + \int_{a}^{b} f(x)dx + \int_{b}^{\infty} f(x)dx$$
$$= \int_{-\infty}^{a} 0dx + \int_{a}^{b} f(x)dx + \int_{b}^{\infty} 0dx$$
$$= \int_{a}^{b} f(x)dx$$

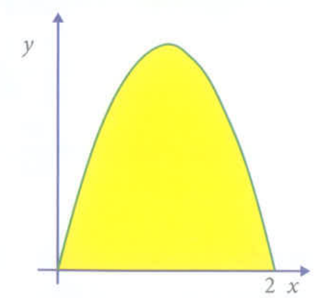
**Example D.8.7**

Find the value of $k$ such that the function:

$$f(x)=\begin{cases} k(2x-x^2) & 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

is a probability density function.

It is generally a good idea to look at these problems graphically. In the present case, the graph is shown. There is a restricted domain. Outside the domain $0 \le x \le 2$, the graph runs along the $x$-axis.



If the function is to be a continuous probability density function, then the shaded area must be 1.

So, we have: $\int_0^2 k\left(2x-x^2\right)dx=1$

$$k\left[\frac{2x^2}{2}-\frac{x^3}{3}\right]_0^2=1$$

$$k\left(4-\frac{8}{3}\right)=1$$

$$k=\frac{3}{4}$$

We are now able to move on to finding probabilities defined by probability density functions.

**Example D.8.8**

The continuous random variable $X$ has a probability density function defined by:

$$f(x)=\begin{cases} k(2-x)^2 & 0\le x\le 2 \\ 0 & \text{otherwise} \end{cases}$$

a Find $k$.

b Find the probability that:

i $0.5 < X \le 1$ ii $X > 1/ X > 0.5$

a As $f(x)$ is a pdf, then:

$$\int_0^2 k(2-x)^2\,dx=1$$

$$k\left[-\frac{1}{3}(2-x)^3\right]_0^2=1$$

$$k\left(0+\frac{8}{3}\right)=1$$

$$k=\frac{3}{8}$$

b i $P(0.5\le X<1)=\int_{0.5}^1 \frac{3}{8}(2-x)^2\,dx$

$$=\left[-\frac{1}{8}(2-x)^3\right]_{0.5}^1$$

$$=\frac{19}{64}$$

ii $P\left(X>1|X>0.5\right)=\dfrac{P(X>1)\cap P(X>0.5)}{P(X>0.5)}$

$$=\frac{P(X>1)}{P(X>0.5)}$$

$P(X>1)=\int_1^2 \frac{3}{8}(2-x)^2\,dx$

$$=\left[-\frac{1}{8}(2-x)^3\right]_1^2$$

$$=\left(0+\frac{1}{8}\right)$$

$$=\frac{1}{8}$$

$P(X>0.5)=\int_{0.5}^2 \frac{3}{8}(2-x)^2\,dx$

$$=\left[-\frac{1}{8}(2-x)^3\right]_{0.5}^2$$

$$=(0+64)$$

$$=\frac{27}{64}$$

$P\left(X>1|X>0.5\right)=\dfrac{1/8}{27/64}=\dfrac{8}{27}$

**Example D.8.9**

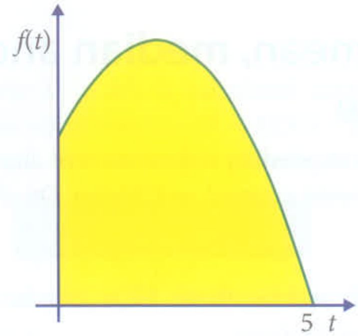
The time ($t$ minutes) that Lennie finds that he has to wait in the supermarket queue before being served is modelled by the function:

$$f(t)=\begin{cases} 0.03\left(5+4t-t^2\right) & 0\le t\le 5 \\ 0 & \text{otherwise} \end{cases}$$

a Sketch the graph of the probability density function.

b Find the probability that Lennie will have to wait between 1 and 3 minutes for service.

a The graph is an inverted parabola. Note again that the domain is defined as the interval [0,5] and that the function is zero elsewhere.




b The required probability is a definite integral. As with other continuous variables, the probability that Lennie will have to wait exactly 3 minutes is zero. It only makes sense to calculate the probability that he will have to wait between two times or less than a given time or more than a given time.

$$\begin{aligned}
P(1 \le T \le 3) &= \int_1^3 0.03\left(5+4t-t^2\right)dt \\
&= 0.03\left[5t+2t^2-\frac{t^3}{3}\right]_1^3 \\
&= 0.03\left(\left(5\times3+2\times3^2-\frac{3^3}{3}\right)-\left(5\times1+2\times1^2-\frac{1^3}{3}\right)\right) \\
&= 0.03\left(15+18-9-5-2+\frac{1}{3}\right) \\
&\approx 0.52
\end{aligned}$$

## Exercise D.8.2

1. Find the value of $k$ such that:

$$f(x)=\begin{cases} kx^2 & 0 \le x \le 3 \\ 0 & \text{otherwise} \end{cases}$$

is a probability density function.

Find also $P(0 \le X \le 1)$.

2. Prove that is a probability density function.

$$f(x)=\begin{cases} 1 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$ . Find $P(0 \le X \le 0.1)$.

3. Find the value of $k$ such that

$$f(x)=\begin{cases} \dfrac{x}{k} & 0 \le x \le 4 \\ 0 & \text{otherwise} \end{cases}$$

is a probability density function. Find $P(2 \le X \le 3)$.

4. Prove that

$$f(x)=\begin{cases} \dfrac{\sin x}{2} & 0 \le x \le \pi \\ 0 & \text{otherwise} \end{cases}$$

is a probability density function. Find also $P(0.5 \le X \le 0.7)$ correct to 3 significant figures.

5. The time ($t$ minutes) between the arrivals of successive buses at a city bus stop is modelled by the function

$$f(t)=\begin{cases} \dfrac{e^{-t/2}}{2} & t \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

a Prove that $f$ represents a probability density function.

b Find the probability that if I have just missed a bus I will have to wait more than ten minutes for the next one.

6. The function $f$ represents the distribution of the amount by which a machine tends to overfill 100 kilogram bags of cement, where $x$ measures the number of kilograms that a bag has been overfilled.

$$f(x)=\begin{cases} \dfrac{k}{x+1} & 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

a Find the value of $k$ such that $f$ represents a probability density function.

b Find the probability that a randomly chosen bag contains more than 101 kg.

7. The time ($t$ minutes) spent by travellers waiting for an urban transit train at a particular station is modelled by the function .

$$f(t)=\begin{cases} 2e^{-2t} & t > 0 \\ 0 & \text{otherwise} \end{cases}$$

a Prove that $f$ represents a probability density function.

b Find the probability that a randomly chosen passenger will have to wait more than 1 minute.

c Find the percentage of passengers that will have to wait more than 2 minutes.

d It is estimated that passengers who have to wait more than 4 minutes at the station will complain to the staff. If the station handles 10000 passengers per day, how many complaints could the staff expect to receive per day?

8. The errors in timing races at the athletics meeting are represented by the function:

$$f(t)=\begin{cases} k\left(1-t^2\right) & -1<t<1 \\ 0 & \text{otherwise} \end{cases}$$

where $t$ is the error in making the measurement with positive $t$ sec representing measured times that were longer than those actually taken and negative values representing measured times that were less than the time actually taken.

a Find the value of $k$ such that $f$ represents a probability density function.

b Find the probability that the error in time measurement of a given race was between 0.1 and 0.5 seconds.

c Find the proportion of the races in which the absolute error in the measurement of the time was less than one tenth of a second.

d The 100 metres sprint was timed at 13.7 seconds. What is the probability that the time actually taken for the race was more than 13.6 seconds?

9. Prove that the function

$$f(x)=\begin{cases} ke^{-kx} & x>0 \\ 0 & \text{otherwise} \end{cases}$$

represents a continuous distribution for all values of $k$ where $k > 0$.

10. Find the exact value of $a$ such that the function

$$f(x)=\begin{cases} x^3 & 0<x<a \\ 0 & \text{otherwise} \end{cases}$$

represents a continuous distribution.

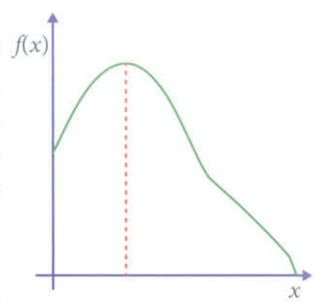
# Mode, mean, median and variance

The mode, mean, median and variance of discrete probability distributions were covered in Chapter D6 of the Common Core Book.

We now look at how these ideas can be transferred to continuous distributions.

## Mode

The mode of a distribution is the value of the variable where the probability density is largest. Graphically, the mode is that value of $x$ which provides the maximum value of the probability density function.




The mode may be found using calculus if the maximum point is not obvious from the graph.

That is, solving the equation: $f'(x)=0$.

## Mean (Expected value)

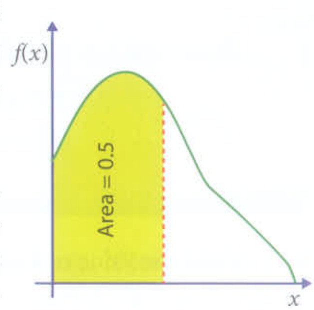
The mean, $\mu = E(X)$, is defined in a way similar to that used in statistics where we calculate $\Sigma x \times f$, the sum of the product of the data values and their frequencies. For a probability function (equivalent to frequency), this becomes:

$$E(X)=\int_{-\infty}^{\infty} x \times f(x)\,dx$$

In practice, the terminals of the integral will be the end points of the domain of the function.

## Median

The median, $m$, is the value of the variable such that half the probability is below that value and half above.




As probability is interpreted as area, this means that we are looking for a value of the variable that has half the area to the left of the value.

That is, we want the value of $m$ such that $\int_{-\infty}^{m} f(x)\,dx=\frac{1}{2}$.

## Variance

The variance, Var($X$) = $\sigma^2$, is calculated using a formula similar to the statistical formula $\sigma^2 = E(X^2) - [E(X)]^2$. For a probability function, the variance is given by:

$$\sigma^2 = \int_{-\infty}^{\infty} x^2 \times f(x)dx - \mu^2$$

The standard deviation, σ, of the distribution is the square root of the variance:

$$\sigma = \sqrt{\sigma^2} = \sqrt{\int_{-\infty}^{\infty} x^2 \times f(x)dx - \mu^2}$$

### Example D.8.10

Find the mode, mean, median, variance and standard deviation of the probability function:

$$f(x) = \begin{cases} \dfrac{6x^2 - 3x^3}{4} & 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

The mode is that value of the random variable that produces the maximum point on the probability function. In this case, we will need to use calculus to find this:

$$f(x) = \frac{6x^2 - 3x^3}{4}$$

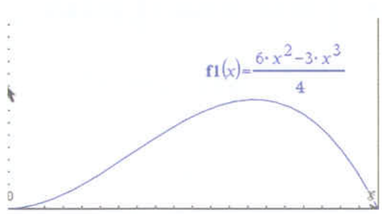
$$f'(x) = 3x - \frac{9}{4}x^2$$

Equation this derivative to zero (to find the maximum point):

$$3x - \frac{9}{4}x^2 = 0$$

$$3x\left(1 - \frac{3}{4}x\right) = 0$$

$$x = 0, \frac{3}{4}$$




A graph will establish which of these values provides that maximum value.

From the graph, it can be seen that the maximum value of the function occurs when $x$ = ¾. We can conclude that the mode is ¾.

The mean is calculated using the formula: $\mu = \int_{-\infty}^{\infty} x \times f(x)dx$

$$E(X) = \int_0^2 x \times \frac{6x^2 - 3x^3}{4}dx$$

$$= \int_0^2 \left(\frac{3}{2}x^3 - \frac{3}{4}x^4\right)dx$$

$$= \left[\frac{3}{8}x^4 - \frac{3}{20}x^5\right]_0^2$$

$$= \frac{3}{8} \times 2^4 - \frac{3}{20} \times 2^5 - 0$$

$$= 1.2$$

This result is a bit less than the mode. Non-symmetric distributions such as this do not necessarily have the same values for mode, mean and median.

Let $m$ be the median of $f(x)$, then, the median satisfies the equation:

$$\int_{-\infty}^{m} f(x)dx = \frac{1}{2}$$

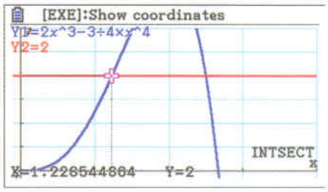
It is now required, is to solve for $m$:

$$\int_0^m \frac{6x^2 - 3x^3}{4}dx = \frac{1}{2}$$

$$\int_0^m \left(6x^2 - 3x^3\right)dx = 2$$

$$\left[2x^3 - \frac{3}{4}x^4\right]_0^m = 2$$

$$2m^3 - \frac{3}{4}m^4 = 2$$




The analytic solution of this equation is beyond the scope of this course so we have used a graphical method to obtain the median $m$ = 1.2285 (to 4 d.p).

Finally, to calculate the variance (and the standard deviation) of the function, we must evaluate:

$$\sigma^2 = \int_{-\infty}^{\infty} x^2 \times f(x)dx - \mu^2$$

$$= \int_0^2 x^2 \times \frac{6x^2 - 3x^3}{4}dx - \mu^2$$

$$= \int_0^2 \frac{6x^4 - 3x^5}{4}dx - 1.2^2$$

$$= \left[\frac{3x^5}{10} - \frac{x^6}{8}\right]_0^m - 1.2^2$$

$$= 1.6 - 1.2^2$$

$$= 0.16$$

Therefore, the variance, Var($X$) = 0.16.

The standard deviation is √0.16 = 0.4

**Example D.8.11**

A continuous random variable $X$ has a probability density function defined by

$$f(x)=\begin{cases} ax^k & x\in[0,2] \\ 0 & \text{otherwise}\end{cases}$$

$a$ and $k$ are positive real constants. Find, in terms of $k$, the expected value and the median of $X$.

As $f(x)$ is a pdf: $\int_{-\infty}^{\infty} f(x)dx=1$.

$$\int_0^2 ax^k dx=1$$

$$a\left[\frac{x^{k+1}}{k+1}\right]_0^2=1$$

$$a\left(\frac{2^{k+1}}{k+1}-0\right)=1$$

$$a=\frac{k+1}{2^{k+1}}$$

$$E(X)=\int_0^2 x\times f(x)dx$$

$$=\int_0^2 ax^{k+1}dx$$

$$=a\left[\frac{x^{k+2}}{k+2}\right]_0^2$$

$$=a\left(\frac{2^{k+2}}{k+2}-0\right)$$

$$=\frac{k+1}{2^{k+1}}\times\frac{2^{k+2}}{k+2}$$

$$=2\left(\frac{k+1}{k+2}\right)$$

Next, the median, $m$, is found by solving $\int_{-\infty}^{m} f(x)dx=\frac{1}{2}$

$$\int_0^m ax^k dx=a\left[\frac{x^{k+1}}{k+1}\right]_0^m$$

$$=a\left(\frac{m^{k+1}}{k+1}-0\right)$$

$$=\left(\frac{k+1}{2^{k+1}}\right)\left(\frac{m^{k+1}}{k+1}\right)$$

$$=\left(\frac{m}{2}\right)^{k+1}$$

Solving for $m$: $\left(\frac{m}{2}\right)^{k+1}=\frac{1}{2}$

$$\frac{m}{2}=\sqrt[k+1]{\frac{1}{2}}$$

$$m=2\times\sqrt[k+1]{\frac{1}{2}}$$

$$=2\times 2^{\left(-1/_{k+1}\right)}$$

$$=2^{\left(k/_{k+1}\right)}$$

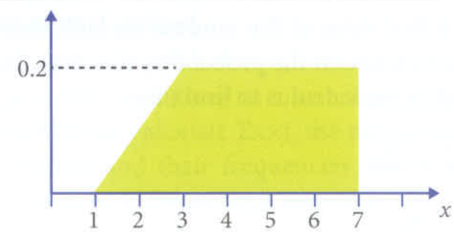
That is, the median, $m=2^{\left(k/_{k+1}\right)}$.

**Example D.8.12**

A continuous random variable $X$ has a probability density function defined by

$$f(x)=\begin{cases} 0.1x-0.1 & 1\le x\le 3 \\ 0.2 & 3\le x\le 7 \\ 0 & \text{otherwise}\end{cases}$$

Sketch the graph of the function. Find the expected value and the median of $X$.




The function is a probability density function because the area is:

$$A=\int_1^3(0.1x-0.1)dx+\int_3^7(0.2)dx$$

$$=\left[\frac{0.1x^2}{2}-0.1x\right]_1^3+[0.2x]_3^7$$

$$=\left(\frac{0.1\times 3^2}{2}-0.1\times 3\right)-\left(\frac{0.1\times 1^2}{2}-0.1\times 1\right)+0.2\times 7-0.2\times 3$$

$$=0.45-0.3-0.05+0.1+1.4-0.6$$

$$=1$$

The median can be found by calculating half of the area working from the right. This is easier than working from the left. The height of the rectangle is 0.2 so we need $0.2x = 0.5$ downwards from 7. So $x = 2.5$ and the median is $7 - 2.5 = 4.5$

The mean is

$$E(X)=\int_1^3 (0.1x-0.1)x\,dx+\int_3^7 (0.2x)dx$$
$$=\left[\frac{0.1x^3}{3}-\frac{0.1x^2}{2}\right]_1^3+\left[0.1x^2\right]_3^7$$
$$=\left(\frac{0.1\times 3^3}{3}-\frac{0.1\times 3^2}{2}\right)-\left(\frac{0.1\times 1^3}{3}-\frac{0.1\times 1^2}{2}\right)+0.1\times 7^2-0.1\times 3^2$$
$$=0.9-0.45-\frac{1}{30}+0.05+4.9-0.9$$
$$=4\frac{7}{15}$$

### Exercise D.8.3

1. For the probability function:

$$f(x)=\begin{cases} 1 & 0\le x\le 1 \\ 0 & \text{otherwise}\end{cases}, \text{ find:}$$

   a the mean and median.

   b the variance and standard deviation.

2. For the probability function

$$f(x)=\begin{cases} 3x^2 & 0\le x\le 1 \\ 0 & \text{otherwise}\end{cases}, \text{ find:}$$

   a The mode, mean and median.

   b The variance and standard deviation.

3. The weights, $w$ grams, of a species of mollusc are distributed according to the function

$$f(x)=\begin{cases} \dfrac{3\left(-24+10x-x^2\right)}{4} & 4\le x\le 6 \\ 0 & \text{otherwise}\end{cases}$$

   a Calculate the mode, mean and median weights.

   b Find the standard deviation of the weights.

   c Assuming that approximately 95% of the weights will lie within 2 standard deviations of the mean, find this 95% confidence interval for the weights of this species of mollusc.

4. The time, $t$ sec, taken to test an electronic circuit is a variable distributed according to the function

$$f(t)=\begin{cases} \dfrac{t}{2} & 0\le t\le 2 \\ 0 & \text{otherwise}\end{cases}$$

   a Calculate the mode, mean and median times.

   b Find the standard deviation of the times.

   c If one of the measures of central tendency was to be used to estimate the amount of time that it would take to test ten thousand of these circuits, which measure would give the best estimate.

5. The time interval, $t$ seconds, between the arrivals of customers at a large supermarket is a continuous random variable modelled by the function:

$$f(t)=\begin{cases} 0.2e^{-0.2t} & t\ge 0 \\ 0 & \text{otherwise}\end{cases}$$

   a Find the modal time between arrivals.

   b Find the median time between customer arrivals.

   c Find the mean time between arrivals. Use an approximate method of integration.

   d Find the standard deviation of the times between arrivals, using an approximate method of integration.

   e Using an approximate 95% confidence interval, what is the longest time between arrivals of customers that is likely to occur?

6. Find the mode, mean and median of the probability distribution:

$$f(x)=\begin{cases} \dfrac{(1-\cos x)}{2\pi} & 0\le x\le 2\pi \\ 0 & \text{otherwise}\end{cases}$$

7. Find the mean and variance of the lengths (cm) of the tails of a species of bird if these lengths are distributed according to the probability function

$$f(x)=\begin{cases}\dfrac{6\left(-66+17x-x^2\right)}{125} & 6\le x\le 11\\ 0 & \text{otherwise}\end{cases}$$

8. The life span of a species of reptiles has been found to have a probability distribution given by:

$$f(t)=\begin{cases}\dfrac{1}{80}e^{-t/80} & t\ge 0\\ 0 & \text{otherwise}\end{cases}$$

a Find the 25th percentile of the life distribution for this species of reptile.

b Find an expression for $F(t) = P(T \le t)$, for $t \ge 0$. $F(t)$ is known as the cumulative distribution of the random variable $T$.

c Sketch the graph of $F(t)$.

9. The random variable X has a probability density function given by

$$f(x)=\begin{cases}ax(b-x) & x\in[0,2]\\ 0 & \text{otherwise}\end{cases}$$

where $a$ and $b$ are positive constants.

a Show that: i $b \ge 2$ ii $a=\dfrac{3}{6b-8}$.

b If $E(X)=\dfrac{8}{7}$, find $a$ and $b$.

c Find the mode of $X$.

10. The time, $t$ days, until recovery after a certain medical procedure is a continuous random variable having a probability density function

$$f(t)=\begin{cases}\dfrac{k}{\sqrt{t-2}} & 3\le t\le 7\\ 0 & \text{otherwise}\end{cases}$$

a Find $k$.

b Find the probability that it will take a patient at least 5 days to recover.

c What is the median recovery time for patients undergoing this procedure?

d Find the expected recovery time, giving your answer to two decimal places.

11. a Differentiate the function $h(x)=x(1-x)^{3/2}$.

b The random variable $X$ has a probability density function:

$$f(x)=\begin{cases}a\sqrt{1-x} & x\in[0,1]\\ 0 & \text{otherwise}\end{cases}$$

where $a$ is a positive real constant.

i Find the value of $a$.

ii Find the median and mode of $X$.

iii Find the exact value of the mean of $X$.

12. Find the mean and variance of these distributions:

a $$f(x)=\begin{cases}0.1 & 0\le x\le 2\\ 0.2 & 3\le x\le 7\\ 0 & \text{otherwise}\end{cases}$$

b $$f(x)=\begin{cases}x^2 & 0\le x\le 1\\ \dfrac{1}{3} & 2\le x\le 4\\ 0 & \text{otherwise}\end{cases}$$

c $$f(x)=\begin{cases}\dfrac{1}{x^2} & 1\le x\le 2\\ x-2 & 2\le x\le 3\\ 0 & \text{otherwise}\end{cases}$$

# Scaling Variables

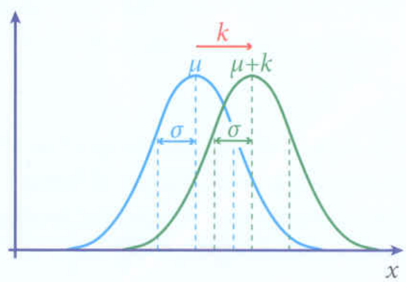
There are also situations in which we need to transform a variable that is the basis of a distribution. This may be because we have made a systematic error in collecting the data. If, for example, when collecting numeric data, we write down numbers that are all 10 too big, the calculated mean will be 10 too big but the spread ($\sigma$) will be correct. What about other possibilities?

What if the data has been recorded in centimetres, but we want all future discussions to be conducted in millimetres?

We will focus on normally distributed variables but the argument applies to other types of distributions.

## Translations

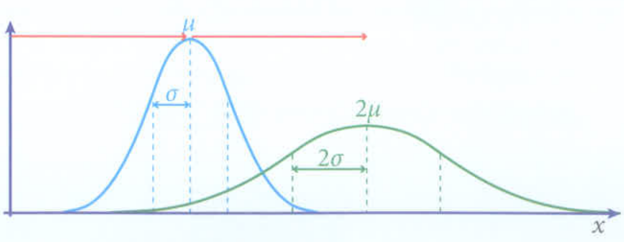
Suppose we have data on the distribution of the heights of a group of people. Graphically it will look a bit like this:




If everybody puts on a party hat $k$ units high then the entire distribution is slid $k$ units to the right. The shape of the distribution is unchanged. The mean will be increased by $k$ units but the standard deviation will be unchanged.

## Dilations

What if, instead, by some magical process everybody's height is doubled? This time the entire curve will be dilated by a factor of 2 from the vertical axis. As we must preserve the unit area, it will also need to be squashed to half its original height.




This time, both the mean and standard deviation will be doubled.

**Example D.8.13**

A normally distributed variable is measured in centimetres and has a mean of 12 cm and a standard deviation of 2 cm. Find the mean and standard deviation if:

a it is found that all the measurements have been recorded as 3 cm too small.

b the variable is converted to millimetres.

a This is a translation of +3. The mean becomes 15 cm and the standard deviation is unchanged at 2 cm.

b This is a dilation by factor 10. The mean becomes 120 (mm) and the standard deviation 20 mm.

**Example D.8.14**

Temperature records in Utopia indicate that the mean daily maximum in May is 86˚F with a standard deviation of 3˚F. Assuming a normal distribution, what will be the mean of the distribution when expressed in ˚Celsius?

What is the probability that a daily maximum is less than 88˚F?

The conversion formula between the two temperature scales is:

$$C = \frac{5}{9}(F - 32)$$

As discussed above, the mean needs to be modified using both the dilation and the translation. Essentially, all we are doing is converting it from ˚F to ˚C.

Mean: $\mu_C = \frac{5}{9}(\mu_F - 32) = \frac{5}{9}(86 - 32) = 30$

The standard deviation in ˚F = 3. What matters here is not the translation of –32 but the scale factor of $^5/_9$$^{\text{ths}}$. The Celsius temperature scale is compressed with respect to the Fahrenheit scale and so the standard deviation must also be compressed.

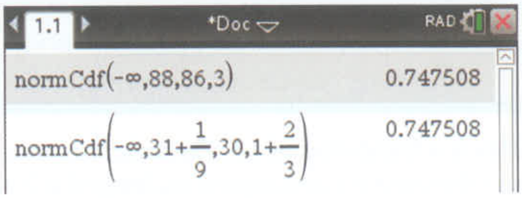
Standard deviation in Celsius is $\sigma_C = \frac{5}{9} \times 3 \approx 1.66667$.

The Celsius distribution is normal with a mean of 30 and a standard deviation of 1.66667.

If this is true, then we should get the same answer to the final calculation if we work in either scale:

In Fahrenheit we are after $T_F < 88$ with $\mu = 86$ and $\sigma = 3$.

In Celsius $C = \frac{5}{9}(88-32) = 31\frac{1}{9}$ $T_C < 31\frac{1}{9}$ with $\mu = 30$ and $\sigma = 1\frac{2}{3}$.

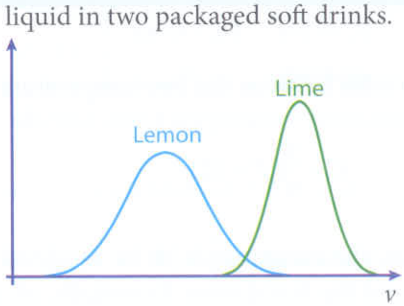



Either way, the probability is 0.748.

## Exercise D.8.4

1. In an experiment, data is collected relating to the amount of heat released in a chemical reaction. The results have a mean of 27.8 kiloCalories with a standard deviation of 2.9 kiloCalories. The results are converted to kiloJoules. Find the mean and standard deviation of the results in kiloJoules. 1 kiloCalorie is equal to 4.2 kiloJoules.

2. The diagram shows two normally distributed variable that represent the distributions of the amounts of liquid in two packaged soft drinks.




Sketch a graph that shows the distribution of the total of liquid in one Lemon and on Lime drink.

3. A variable $X$ with a mean of 2.6 and a standard deviation of 0.3 is transformed using $Y = 2X - 1$. Find the mean and standard deviation of $Y$.

4. The speeds of cars passing a checkpoint were recorded. The mean speed was 65 mph with a standard deviation of 3 mph. What will be these statistics if the speeds are measured in kph? 5 mph ≈ 8 kph.

5. An electrical component is composed of these parts.

| Number used | Description | Mean (gms) (each) | St. Dev (gms) (each) |
|---|---|---|---|
| 1 | Casing | 28 | 0.3 |
| 6 | Screws | 0.8 | 0.05 |
| 1 | Board | 23 | 0.9 |
| 4 | Wires | 1.5 | 0.02 |

Find the mean and standard deviation of the weight of the complete assembly.

6. The rubber tubes we occasionally see nailed across roads are there to take automatic traffic censuses. If there are a pair of them, it is usually because the survey measures speed (from the time interval between 'hits' on the two tubes). In a study in which the two tubes were 1 metre apart, the mean speed of the vehicles that passed the survey point was 68 kph with a standard deviation of 5 kph.

   a. Find the time taken for a car travelling at the mean speed to pass from the first to the second tube.

   After the survey, it was discovered that the tubes had been incorrectly placed. Instead of being 1 metre apart, the actual distance was 90 cm. The timing equipment was, however, working correctly.

   b. Use your result from part a to calculate the correct speed to pass between the actual positions of the tubes in the measured time calculated in a.

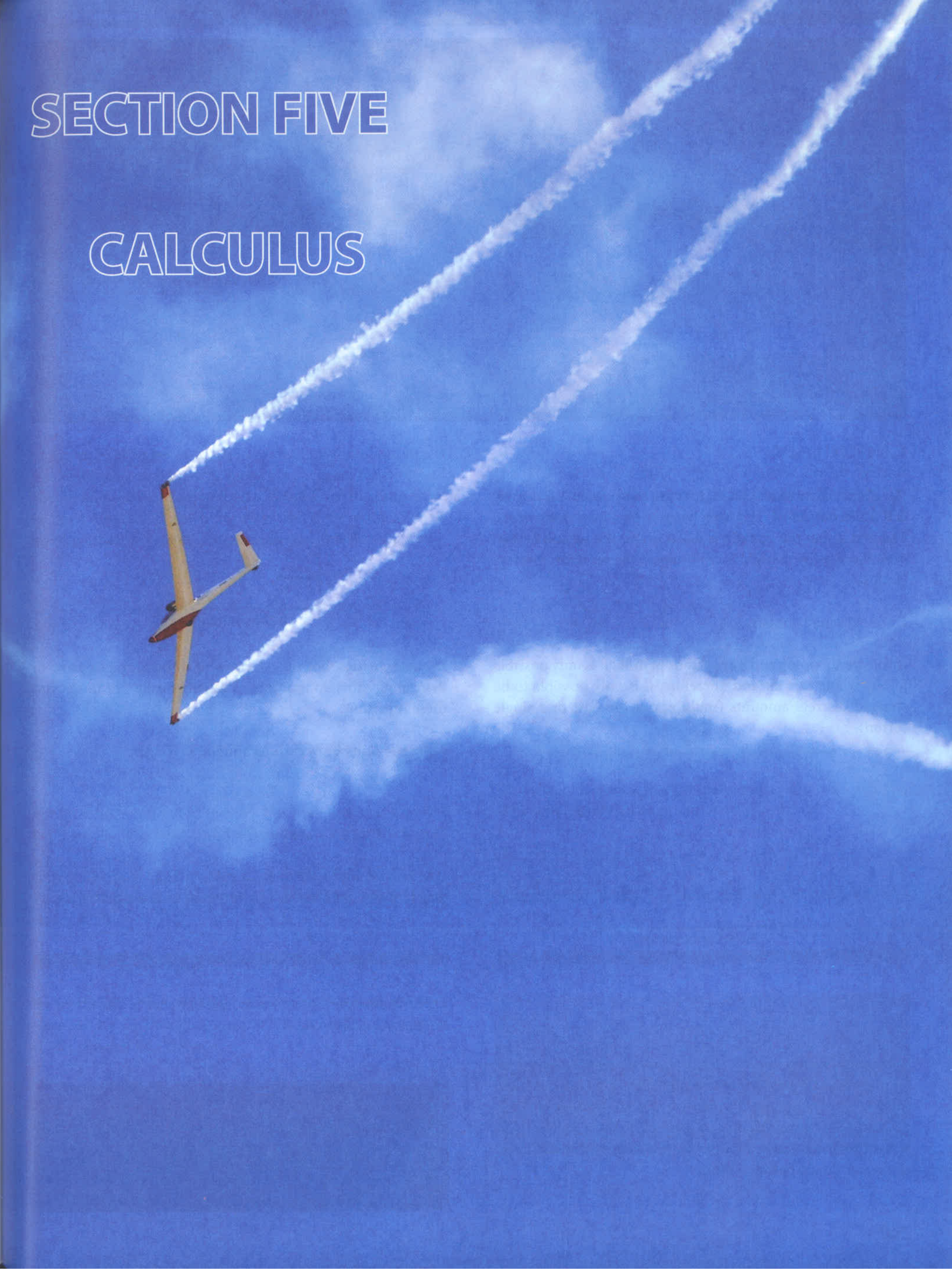
   c. Find the corrected mean and standard deviation speeds for the survey.

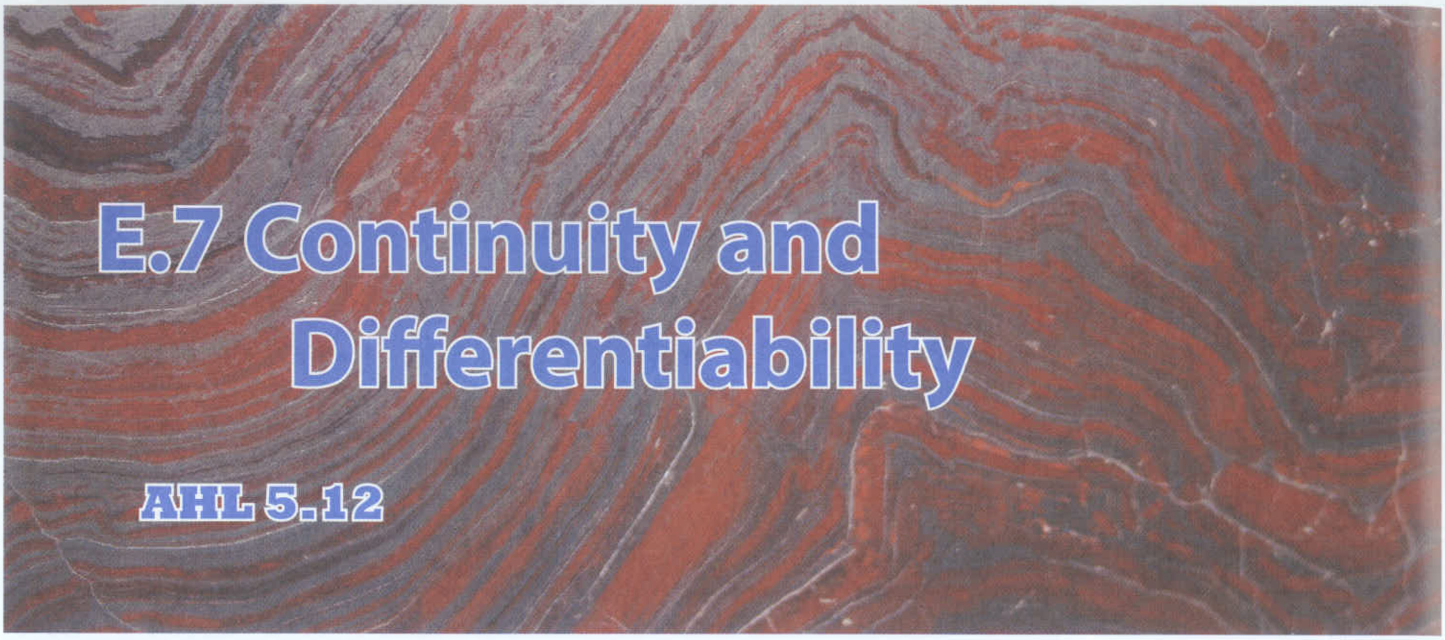
7. The binomial sum variance inequality states that the variance of the sum of binomially distributed random variables will always be less than or equal to the variance of a binomial variable with the same $n$ and $p$ parameters. Can you prove this?

Answers

# SECTION FIVE

# CALCULUS





# E.7 Continuity and Differentiability

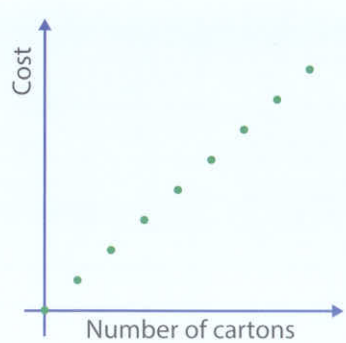
AHL 5.12

## Continuity

Many of the functions that we use in everyday life display discontinuity. This can happen in a two main ways. The first is that the domain is discrete and the second is that, even though the domain is continuous, the function displays 'jumps'.

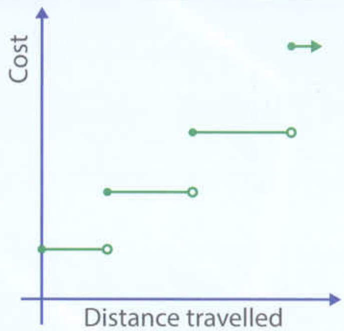
1. Discrete domain

If you are shopping and buy cartons of milk, it is only possible to buy whole cartons and, therefore, it is only possible to be charged discrete amounts (multiples of the cost of a single carton).




2. Jump discontinuities

When travelling in a taxi, it is common for the charge to be 'metered'. Usually, this is done by an electronic device that continuously displays the charge to the customer. In a simple model, this depends on distance travelled, though most taxis also have a 'time taken' component in their charges. The point is, however, that only certain levels of charge are possible and the meter jumps between them.




In mathematics, both these types of discontinuity occur. In describing our two examples mathematically, we might use these functions:

Milk: $n$ = number of cartons bought, $C$ = cost:

$$C = 1.27n,\ n = 0, 1, 2, 3, \ldots$$

Taxi: $d$ = distance travelled, $C$ = cost:

$$C = \begin{cases} 5.70 & 0 \le d < 3 \\ 7.65 & 3 \le d < 8 \\ 9.20 & 8 \le d < 11 \\ 11.55 & 11 \le d < \ldots \end{cases}$$

There is, however, a further possibility that is illustrated by the following example.

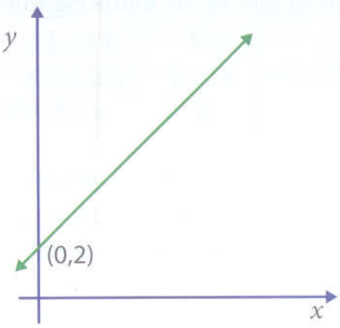
**Example E.7.1**

Sketch the graph of $y = \dfrac{x^2 - 4}{x - 2}$

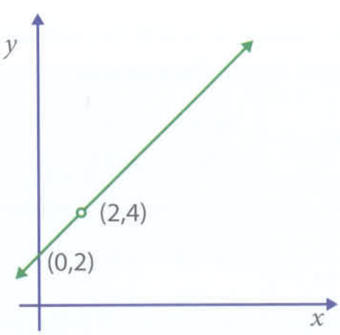
This seems to be made much simpler by this piece of algebra:

$$y=\frac{x^2-4}{x-2}=\frac{(x+2)(x-2)}{x-2}=x+2$$

This suggests that the graph is:




But hold on! The step: $\frac{(x+2)(x-2)}{x-2}=x+2$ involves division of the numerator by $x-2$. This is zero if $x=2$ and the step is not legitimate. Alternatively, look at the original function and you should see that 2 should be missing from the domain as it, again, implies division by zero. Thus, the point (2,4) is missing from the graph. We show this in the following way.




We have little option but to show this puncture as if it were a gaping hole - else how would we see it?

In fact, the hole has no width at all as the single number 2 is missing. If this were a puncture in a motor tyre, it would leak no air at all.

In addition to the 'puncture' discontinuity of the previous example, we also need to remember the asymptotic behaviour of the rational functions.

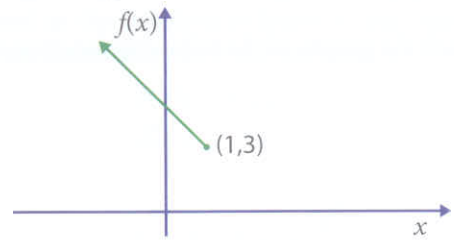
Review Chapter B6 of this book to remind yourself of these.

**Example E.7.2**

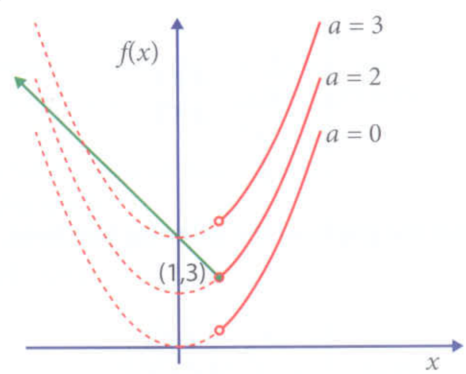
For what value of $a$ will the function:

$$f(x)=\begin{cases} x^2+a & x>1 \\ -x+4 & x\le 1 \end{cases}$$ be continuous?

This hybrid has two parts. The second part is a single function:




The first part is a family of functions that are all vertical transformations of the same parabolic graph. However, the constraint $x>1$ means that all the graph to the left of $x=1$ is removed.




The value of $a$ is such that the green graph intersects with the red graph (family) at $x=1$.

$$1^2+a=-1+4\Rightarrow a=2$$

Note that the green 'dot' now fills the red 'ring' and the function is continuous.

**Example E.7.3**

For what value of $b$ will the function:

$$f(x)=\begin{cases} -x+4 & x>3 \\ 2-\sqrt{x-b} & x\le 3 \end{cases}$$ be continuous?

This time we are trying to make the line $y=-x+4$ meet the curve $y=2-\sqrt{x-b}$ at $x = 3$.

Thus: $-3+4=2-\sqrt{3-b}$

$$-1=-\sqrt{3-b}$$
$$1=3-b$$
$$b=2$$

## Exercise E.7.1

1. Sketch the graphs of the following functions:

a $f(x)=\begin{cases} -x+2 & x>3 \\ 1 & x\le 3 \end{cases}$

b $f(x)=\begin{cases} x^2+2 & x>-1 \\ 3 & x\le -1 \end{cases}$

c $f(x)=\begin{cases} \sqrt{x} & x\ge 4 \\ 6-x & x<4 \end{cases}$

d $f(x)=\begin{cases} \dfrac{1}{x} & x\ge 1 \\ 2-x^2 & x<1 \end{cases}$

2. Sketch the graphs of the following functions:

a $f(x)=\begin{cases} -2 & x<0 \\ x-2 & 0\le x\le 4 \\ 2 & x>4 \end{cases}$

b $h(x)=\begin{cases} \sqrt{-x} & x\le -1 \\ 1-x^2 & -1<x<1 \\ -\sqrt{x} & x\ge 1 \end{cases}$

3. Sketch the graphs of the following functions:

a $h(x)=\begin{cases} x^3+1 & x>0 \\ -1 & x\le 0 \end{cases}$

b $g(x)=\begin{cases} x+2 & x>1 \\ x^2-1 & x\le 1 \end{cases}$

c $f(x)=\begin{cases} \dfrac{x}{x+1} & x\ge 0 \\ 1 & x<0 \end{cases}$

d $f(x)=\begin{cases} 2-\sqrt{x} & x>0 \\ x+3 & x\le 0 \end{cases}$

4. Sketch the graphs of the following functions:

a $f(x)=\begin{cases} -4 & x<-2 \\ x^2-4 & -2\le x\le 2 \\ 4 & x>2 \end{cases}$

b $h(x)=\begin{cases} \sqrt{2-x} & x\le -2 \\ -2x & -2<x<2 \\ -\sqrt{x+2} & x\ge 2 \end{cases}$

5. Sketch the graphs of the following functions:

a $f(x)=\begin{cases} \dfrac{1}{x+1} & x\ge 0 \\ a & x<0 \end{cases}, a>1$

b $f(x)=\begin{cases} a+x^2 & x>0 \\ x+3 & x\le 0 \end{cases}, a<-2$

6. For what value(s) of $a$ will the following functions be continuous? Sketch their graphs.

a $f(x)=\begin{cases} ax+1 & x>1 \\ 5 & x\le 1 \end{cases}$

b $h(x)=\begin{cases} 2x-4 & x\ge 2 \\ a-2x & x<2 \end{cases}$

c $h(x)=\begin{cases} ax^3-1 & x>2 \\ 3+x^2 & x\le 2 \end{cases}$

d $h(x)=\begin{cases} \dfrac{1}{a}x^2+1 & x\ge 2 \\ ax+1 & x<2 \end{cases}$

7. Given that $-\dfrac{1}{2}\le\dfrac{x}{x^2+1}\le\dfrac{1}{2}$, sketch the graph of:

$f(x)=\dfrac{2ax}{x^2+1}$ for $x\in(-\infty,\infty)$ where $a > 0$. For what value(s) of $a$ will the function $h(x)=\begin{cases} \dfrac{2ax}{x^2+1} & x\ge 1 \\ 4 & x<1 \end{cases}$ be continuous? Sketch the graph of $h$.

## Differentiability

Functions are differentiable at a point in their domain if the limit:

$$\lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$$

is properly defined.

By 'properly defined', we mean two things:

1. All the terms in the expression must exist.

This implies that, if a function has a discontinuity ($f(x)$ not defined), it cannot have a derivative (or gradient) at the discontinuity. We cannot draw a tangent at a point that does not exist.

2. The limit must be the same if evaluated from the left (negative values of $h$) or from the right (positive values of $h$).

This rather more complex criterion is illustrated in the next example.

**Example E.7.4**

Investigate the differentiability of these two functions at the point where $x = 2$.

$$f(x)=\begin{cases} x^2 & x<2 \\ x+2 & x\geq 2\end{cases} \qquad g(x)=\begin{cases} x^2 & x<2 \\ 4x-4 & x\geq 2\end{cases}$$

a $$f(x)=\begin{cases} x^2 & x<2 \\ x+2 & x\geq 2\end{cases}$$

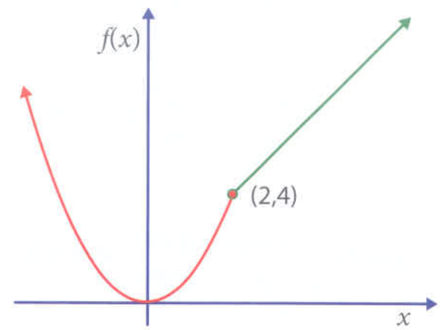
If we look at the gradient limit approaching $x = 2$ from the left, we must use the first part of the rule to evaluate the $f(x + h)$ part of the expression ($h$ is negative). However, $f(2) = 4$ must come from the second part of the rule:

$$\lim_{h\to 0}\left(\frac{f(2+h)-f(2)}{h}\right)=\lim_{h\to 0}\left(\frac{(2+h)^2-4}{h}\right)$$
$$=\lim_{h\to 0}\left(\frac{4+4h+h^2-4}{h}\right)$$
$$=\lim_{h\to 0}\left(\frac{4h+h^2}{h}\right)$$
$$=\lim_{h\to 0}(4+h)$$
$$=4$$

If we look at the limit approaching $x = 2$ from the right ($h$ positive), all parts of the limit expression from the second part of the rule:

$$\lim_{h\to 0}\left(\frac{f(2+h)-f(2)}{h}\right)=\lim_{h\to 0}\left(\frac{(2+h)+2-4}{h}\right)$$
$$=\lim_{h\to 0}\left(\frac{h}{h}\right)$$
$$=1$$

We are being a bit pedantic in using a limiting process to find this gradient as the function is linear to the right of $x = 2$, but it is what should happen if the question is to be answered rigorously. Graphically, this is:




There is a corner at the point (2,4). This is why the tangent is not properly defined. The function is not differentiable at $x = 2$ but is differentiable eveywhere else.

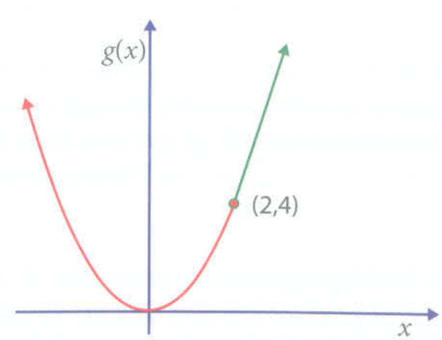
Whilst we have used a limiting argument, the rules of calculus confirm this result:

For the red curve: $f(x)=x^2$

$$f'(x)=2x$$
$$f'(2)=2\times 2$$
$$=4$$

For the green line, the gradient is 1, confirming that the gradients of the two parts of the curve are different and that there is a 'corner'.

For $g(x)$, the gradients of the two parts are both 4 at (2,4). The function is differentiable everywhere.



## Exercise E.7.2

1. State whether or or not the following functions have derivatives at the $x$ values given.

   a $f(x)=\begin{cases} x^2 & x<1 \\ x^3 & x\geq 1 \end{cases}$ at $x = 1$.

   b $f(x)=\begin{cases} 3x^2 & x<1 \\ 4-x^3 & x\geq 1 \end{cases}$ at $x = 1$.

   c $f(x)=\begin{cases} 3x^2 & x<1 \\ 2x^3+1 & x\geq 1 \end{cases}$ at $x = 1$.

   d $f(x)=\begin{cases} xe^x & x<0 \\ x & x\geq 0 \end{cases}$ at $x = 0$.

   e $f(x)=\begin{cases} x\sin x & x<0 \\ x & x\geq 0 \end{cases}$ at $x = 0$.

   f $f(x)=\begin{cases} \sin x & x<0 \\ \cos(x+\pi) & x\geq 0 \end{cases}$ at $x = 0$.

2. Find the values of $a$ and $b$ such that the function:

   $$f(x)=\begin{cases} x^2+ax+b & x<1 \\ 4x-4 & x\geq 1 \end{cases}$$

   is both continuous and differentiable at the point (1,0).

3. Find the values of $a$ and $n$ such that the function:

   $$f(x)=\begin{cases} \sin(nx) & x<0 \\ ax & x\geq 0 \end{cases}$$

   is both continuous and differentiable at the point (1,0).

4. Prove that the function $f(x)=\left|e^{2x}\right|, x\in\mathbb{R}$ is both continuous and differentiable throughout its domain. Can the same be said of: $g(x)=e^{2|x|}, x\in\mathbb{R}$ ?

5. Use a limiting argument to prove that if $f(x)=e^x$, then $f'(x)=e^x$. What does this say about the result of differentiating the function a second time.

6. The parabola: $f(x)=ax^2+bx+c, x\leq 1$ passes through the points (0,2), (–1,1) & (–3,–7).

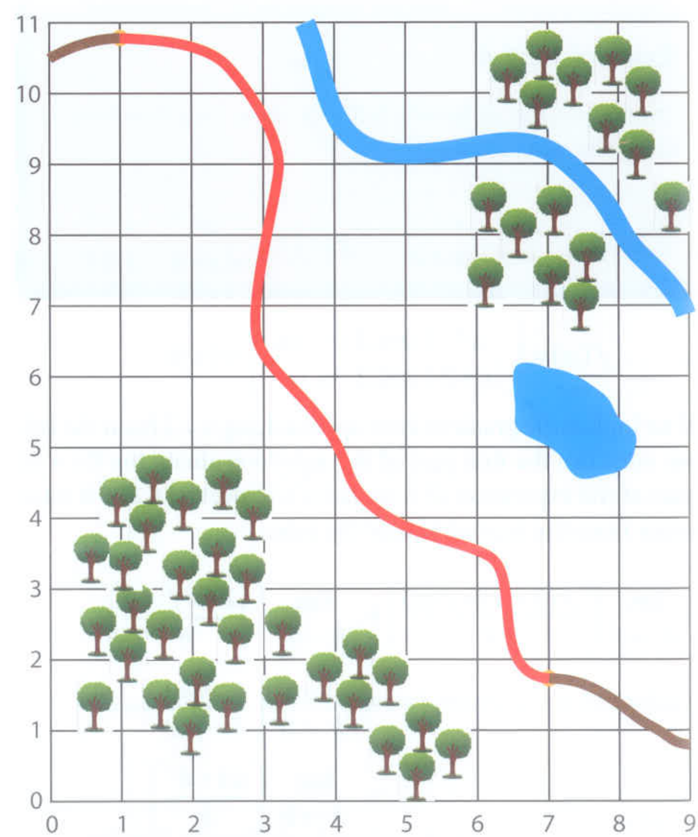
   The parabola: $g(x)=px^2+qx+r, x>1$ passes through the points (2,0), (3,1) & (4,4).

   Prove that the function defined by $f\cup g$ is both continuous and differentiable for $x\in\mathbb{R}$.

7. Investigate the continuity and differentiability of thr family of functions:

   $$f(x)=\frac{x^2+a}{x+a}, x\in\mathbb{R}, a\in\mathbb{Q}.$$

8. The map shows an old and rather winding road. It is desired to replace the red section of the road with a smoother version. The brown sections are to remain. The replacement road meets both of these in an east-west direction. Find a cubic polynomial that meets the parts of the road that are to remain in a smooth manner and will provide a suitable replacement road.



# Higher Derivatives

Since the derivative of a function $f$ is another function, $f'$, then it may well be that this derived function can itself be differentiated. If this is done, we obtain the second derivative of $f$ which is denoted by $f''$ and read as "$f$–double–dash".

The following notation for $y = f(x)$ is used:

First derivative $\quad \frac{dy}{dx} = f'(x)\ [= y']$

Second derivative $\quad \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = f''(x)\ [= y'']$

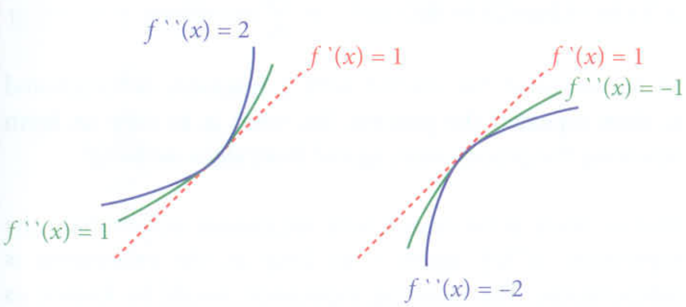
So, for example, if $f(x) = x^3 - 5x^2 + 10$

then $f'(x) = 3x^2 - 10x$ and $f''(x) = 6x - 10$.

The expression $\frac{d^2y}{dx^2}$ is read as "dee–two–$y$ by dee–$x$–squared" and the expression $y''$ is read as "$y$–double–dash".

It is important to have some picture of the meaning of a second derivative as it is reflected in the shapes of graphs.

The second derivative measures the rate of change of the gradient of a curve. In general terms, this is often described as 'curvature'. This is a rather loose term which does have a precise definition that is not required for this course. A descriptive understanding is, however, useful.




As the diagram shows, if the second derivative is big and positive, the graph 'curls upwards' more rapidly than if it is small and positive. The same applies to the 'downward curl' of graphs with negative second derivatives. The blue lines are more curved than the green lines.

**Example E.7.5**

Find the second derivative of:

a $x^4 - \sin 2x$ b $\ln(x^2+1)$ c $x \times \mathrm{Sin}^{-1}x$

a Let $y = x^4 - \sin 2x$ then $y' = 4x^3 - 2\cos 2x$ and $y'' = 12x^2 + 4\sin 2x$.

b Let $f(x) = \ln(x^2+1)$ then $f'(x) = \frac{2x}{x^2+1}$ and

$$f''(x) = \frac{2(x^2+1) - 2x(2x)}{(x^2+1)^2} = \frac{2-2x^2}{(x^2+1)^2}$$

c Let $y = x\mathrm{Sin}^{-1}x$ then

$$\frac{dy}{dx} = x \times \frac{1}{\sqrt{1-x^2}} + (1) \times \mathrm{Sin}^{-1}x = \frac{x}{\sqrt{1-x^2}} + \mathrm{Sin}^{-1}x$$

$$\frac{d^2y}{dx^2} = \frac{(1)\times\sqrt{1-x^2} - x \times \frac{1}{2}(-2x)\cdot\frac{1}{\sqrt{1-x^2}}}{(\sqrt{1-x^2})^2} + \frac{1}{\sqrt{1-x^2}}$$

$$= \frac{\sqrt{1-x^2} + \frac{x^2}{\sqrt{1-x^2}}}{1-x^2} + \frac{1}{\sqrt{1-x^2}}$$

$$= \frac{\frac{1-x^2}{\sqrt{1-x^2}} + \frac{x^2}{\sqrt{1-x^2}}}{1-x^2} + \frac{1}{\sqrt{1-x^2}}$$

$$= \frac{\frac{1}{\sqrt{1-x^2}}}{1-x^2} + \frac{1}{\sqrt{1-x^2}}$$

$$= \frac{1}{(1-x^2)\sqrt{1-x^2}} + \frac{1}{\sqrt{1-x^2}}$$

$$= \frac{1}{(1-x^2)\sqrt{1-x^2}} + \frac{1-x^2}{(1-x^2)\sqrt{1-x^2}}$$

$$= \frac{2-x^2}{(1-x^2)\sqrt{1-x^2}}$$

As we can see from Example c, some second derivatives require the use of algebra to obtain a simplified answer.

Note then that, just as we can find the second derivative, so too can we determine the third derivative and the fourth derivative and so on (of course, assuming that these derivatives exist). We keep differentiating the results. The notation then is extended as follows:

Third derivative is $f'''(x)$ ("$f$–triple–dash") and so on where the $n$th derivative is $f^{(n)}(x)$ or $\frac{d^ny}{dx^n}$.

## Exercise E.7.3

1. Find the second derivative of the following functions.

   a $f(x) = x^5$

   b $y = (1+2x)^4$

   c $f\colon x \mapsto \frac{1}{x}$ where $x \in \mathbb{R}$

   d $f(x) = \frac{1}{1+x}$

   e $y = (x-7)(x+1)$

   f $f\colon x \mapsto \frac{x+1}{x-2}$ where $x \in \mathbb{R}\backslash\{2\}$

   g $f(x) = \frac{1}{x^6}$

   h $y = (1-2x)^3$

   i $y = \ln x$

   j $f(x) = \ln(1-x^2)$

   k $y = \sin 4\theta$

   l $f(x) = x\sin x$

2. Find the second derivative of the following.

   a $\arctan x$  b $\arcsin x$

   c $\arccos x$  d $x\arctan x$

   e $\arcsin\sqrt{x}$  f $\arccos\left(\frac{1}{\sqrt{x}}\right)$

3. Find the second derivative of the function $f(x) = \frac{\log_e x}{x^2}$

   Find a formula for the second derivative of the function $f(x) = \frac{\log_e x}{x^n}$.

4. Consider the function $f(x) = \frac{1}{x+1}, x \neq -1$.

   Find the first five derivatives by differentiating the function five times. Hypothesise a formula for the $n$th derivative of this function. Use the method of mathematical induction or other appropriate method to prove that your formula works for all whole numbered values of $n$.

5. Find a formula for the second derivative of the family of functions $f(x) = \left(\frac{x+1}{x-1}\right)^n$ where $n$ is a real number.

6. Given $y = \frac{1}{1-x}$, prove that $\frac{d^n y}{dx^n} = \frac{n!}{(1-x)^{n+1}}$ for $n \geq 1$.

Extra questions

Mostly, we have dealt with the concept of 'first derivatives' and 'second derivatives'. However, there is no reason for not having higher order derivatives, and in fact, you might already have done so – for example, in physics.

Consider the displacement function for an object whose position, $x(t)$, at time $t$, $t \geq 0$ is given by $x(t) = 3\sin(2t) + 3t^3$.

Finding the first derivative gives us the rate of change of displacement with respect to time, which is known as the **velocity**. So, finding the first derivative gives:

$$x'(t) = 6\cos(2t) + 9t^2$$

We can then determine the **acceleration** of this object by determining the rate of change of its velocity; that is, we differentiate: $x'(t) = 6\cos(2t) + 9t^2$ giving:

$$x''(t) = -12\sin(2t) + 18t$$

So far, so good. We started with a function, differentiated it, then repeated the process. So, what is to stop us from repeating the process once again? Basically – nothing!

That is, there is no reason why we cannot differentiate the expression $x''(t)$ again – as long as the expression is differentiable. The resulting expression would be known as … yes, you've guessed correctly, the third derivative.

$$x'''(t) = -24\cos(2t) + 18$$

In this case, it measures the rate of change of acceleration.

This process can proceed indefinitely in this case as the functions produced are all differentiable. The only problem is the notation. The fifth derivative would need five dashes so instead of writing $x'''''(t)$, we use the notation $x^{(5)}(t)$.

How do we interpret the 4th derivative, the 5th derivative and so on? Well, basically they are simply the rate of change of the quantity that has just been differentiated. So, the 5th derivative is the rate of change of the 4th derivative (and, if you know what the 4th derivative represents, then you'll know the rate of change of that). Similarly, the 3rd derivative is the rate of change of the 2nd derivative and the argument is the same.

**Example E.7.6**

Find the first four derivatives of the following.

a $f(x)=x^3+e^{-2t}$

b $y=\cos(2t)$

c $h(x)=x\log_e(x), x>0$

a $f(x) = x^3 + e^{-2x} \Rightarrow f'(x) = 3x^2 - 2e^{-2x}$

$\therefore f''(x) = 6x + 4e^{-2x}$

$\therefore f'''(x) = 6 - 8e^{-2x}$

So that the 4th derivative, $f''''(x) = 16e^{-2x}$.

Or, if you prefer, $f^{(4)}(x) = 16e^{-2x}$.

b $y = \cos(2t) \Rightarrow \frac{dy}{dt} = -2\sin(2t)$

$\therefore \frac{d^2y}{dx^2} = -4\cos t(2t)$

$\frac{d^3y}{dx^3} = 8\sin(2t)$

$\therefore \frac{d^4y}{dx^4} = 16\cos(2t)$

c With $h(x) = x\log_e(x), x > 0$, we have:

$h'(x) = 1 \times \log_e(x) + x \times \frac{1}{x} = \log_e(x) + 1$

$\Rightarrow h''(x) = \frac{1}{x}$

$h'''(x) = -\frac{1}{x^2}$

$\therefore h^{(4)}(x) = \frac{2}{x^3}$

**Example E.7.7**

Find the third derivative of:

a $y=\log_e(1+x)$

b $y=\log_e(1+x^2)$

a Starting with $y = \log_e(1+x)$ we have $\frac{dy}{dx} = \frac{1}{1+x}$.

$\frac{d^2y}{dx^2} = -\frac{1}{(1+x)^2}$

$\therefore \frac{d^3y}{dx^3} = \frac{2}{(1+x)^3}$

b Starting with $y = \log_e(1+x^2)$, we have $\frac{dy}{dx} = \frac{2x}{1+x^2}$.

$$\therefore \frac{d^2y}{dx^2} = \frac{2 \times (1+x^2) - 2x \times (2x)}{(1+x^2)^2} = \frac{2+2x^2-4x^2}{(1+x^2)^2} = \frac{2-2x^2}{(1+x^2)^2}$$

Next, 
$$\therefore \frac{d^3y}{dx^3} = \frac{-4x \times (1+x^2)^2 - 2(1-x^2) \times (2 \times 2x(1+x^2))}{(1+x^2)^4}$$
$$= \frac{[-4x(1+x^2) - 2(1-x^2) \times 4x](1+x^2)}{(1+x^2)^4}$$
$$= -\frac{[4x+4x^3+8x(1-x^2)]}{(1+x^2)^3}$$
$$= -\frac{(12x-4x^3)}{(1+x^2)^3}$$
$$= \frac{4x^3-12x}{(1+x^2)^3}$$

The examples above highlight how quickly the work involved grows when dealing with expressions that will require the use of a number of differentiation rules for subsequent derivatives. In example b we started by using the chain rule and log rule, for the second derivative we used the quotient rule and the chain rule (and similarly for the third derivative). This shows how quickly the work involved can grow when determining higher derivatives.

Let's go back to a of the above example. Notice the derivatives are:

$\frac{dy}{dx} = \frac{1}{1+x}$; $\frac{d^2y}{dx^2} = -\frac{1}{(1+x)^2}$ and $\therefore \frac{d^3y}{dx^3} = \frac{2}{(1+x)^3}$.

Had we continued on, we would have found that:

$$\frac{d^4y}{dx^4} = \frac{-2 \cdot 3}{(1+x)^4}; \frac{d^5y}{dx^5} = \frac{2 \cdot 3 \cdot 4}{(1+x)^5} \text{ and so on.}$$

In fact, by observation, we have that:

$$\frac{d^ny}{dx^n} = \frac{(-1)^{n-1} \cdot 2 \cdot 3 \cdot 4 \cdot \ldots \cdot (n-1)}{(1+x)^n}$$

$$= \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}$$

Such a result would need to be proven by the use of mathematical induction, which we will leave for you to do in the exercises that follow.

## Exercise E.7.4

1. Find the fourth derivative of:

   a $g(x) = \sqrt{x} + \frac{2}{x^2}$

   b $k(t) = 3t^3 - t^{-1} + 2$

   c $h(x) = \frac{5}{2x-1}$

   d $f(x) = \cos(3x) + \sin(2x)$

2. Find the third derivative of:

   a $x\cos(x)$

   b $\sin(x)\cos(x)$

   c $\sin^2 2x$

   d $\tan(2x)$

3. For the function $f(x) = \sqrt{x^2+1}$ calculate its fourth derivative at the origin.

4. Evaluate the third derivative of the function $f(x) = x\text{Sin}^{-1}(x)$ at the origin.

5. Evaluate the third derivative, at the origin, of:

   a $2^{\sin(x)}$

   b $\log_3(x+1)$

   c $x \times 3^x$

6. Evaluate $f^{(4)}(1)$ if $f(x) = 4^x$.

Extra questions

Answers


We have been discussing continuity.

The French mathematician Henri Léon Lebesgue (pronounced 'le-baig') (1875–1941), see right, investigated the implications of continuity in considerably more detail than we have here.

His findings are some of the most mysterious and elegant in the whole of mathematics and will repay further investigation.

Just as a taster, consider the function:

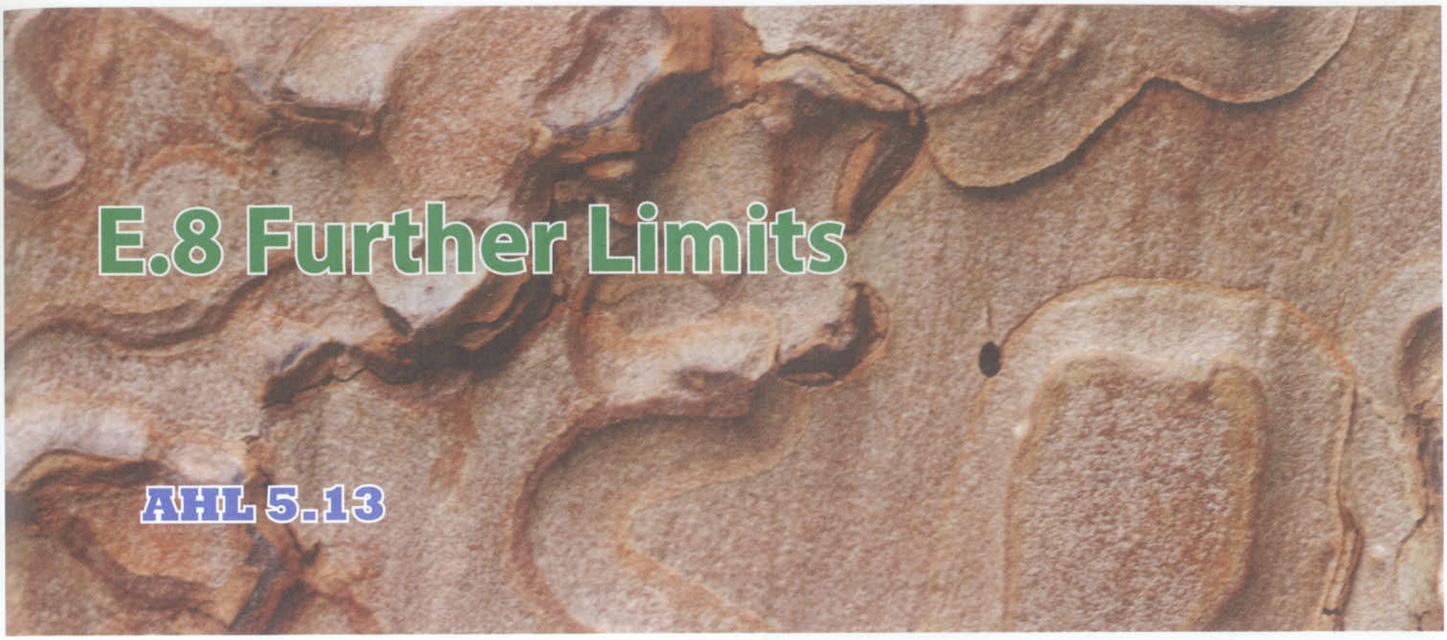
$$f(x) = \begin{cases} 0 & x \text{ rational} \\ 1 & x \text{ irrational} \end{cases}$$

Since there is a rational number between any two irrational numbers and an irrational number between any two rational numbers, this function is discontinuous **everywhere**.

It follows that it is differentiable **nowhere**.

However, one of the most surprising results is that the function is integrable and:

$$\int_0^1 f(x)dx = 1$$



# E.8 Further Limits

AHL 5.13

## L'Hôpital's Rule

French mathematician Guillaume François Antoine, Marquis de l'Hôpital (1661 – 1704) is chiefly remembered for a limits rule that bears his name. The name is also frequently spelled l'Hôspital.

L'Hôpital's Rule is particularly useful in evaluating limits that involve expressions that resolve to $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

L'Hôpital's Rule is usually stated as:

If $\lim\limits_{x\to c}\frac{f(x)}{g(x)}$ takes the indeterminate form $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$, then:

$$\lim_{x\to c}\frac{f(x)}{g(x)}=\lim_{x\to c}\frac{f'(x)}{g'(x)}$$

The full proof of this result is quite complex. We will show that the result holds true for the indeterminate form when $f(c)=g(c)=0$.

$$\lim_{x\to c}\frac{f(x)}{g(x)}=\lim_{x\to c}\frac{f(x)-0}{g(x)-0}$$

$$=\lim_{x\to c}\frac{f(x)-f(c)}{g(x)-g(c)}$$

$$\lim_{x\to c}\frac{f(x)}{g(x)}=\frac{\lim\limits_{x\to c}\frac{f(x)-f(c)}{x-c}}{\lim\limits_{x\to c}\frac{g(x)-g(c)}{x-c}}$$

$$=\frac{\lim\limits_{x\to c}f'(x)}{\lim\limits_{x\to c}g'(x)}$$

$$=\lim_{x\to c}\frac{f'(x)}{g'(x)}$$

So, as long as $g'(c)\neq 0$, the result is complete.

If the quotient of the derivatives is still of the form $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$ we have to apply L'Hopital's rule again and calculate the quotient of the second, third,.... derivatives at $x = c$ until the quotient yields a properly defined value.

The first of our examples deals with a very important limit that is crucial in the first principles differentiation of the trigonometric functions.

**Example E.8.1**

Use L'Hôpital's Rule to evaluate: $\lim\limits_{x\to 0}\frac{\sin(x)}{x}$

As $\frac{\sin(0)}{0}$ is of the form $\frac{0}{0}$ we can apply L'Hopital's Rule.

Letting: $f(x)=\sin(x), g(x)=x$ , we use calculus:

$f'(x)=\cos(x), g'(x)=1$

$$\lim_{x\to 0}\frac{\sin(x)}{x}=\lim_{x\to 0}\frac{\cos(x)}{1}$$
$$=\frac{\cos(0)}{1}$$
$$=1$$

### Example E.8.2

Evaluate: $\lim_{x\to 1}\frac{\ln(x)}{x-1}$

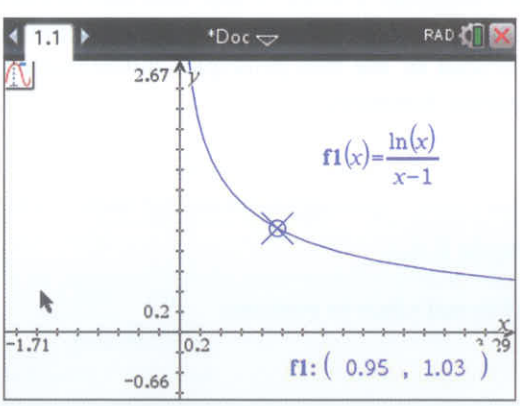
Let: $f(x)=\ln(x), g(x)=x-1$

$f'(x)=\frac{1}{x}, g'(x)=1$

$f(1)=\ln(1)=0, g(1)=1-1=0$ and so L'Hopital's rule is applicable.

$$\lim_{x\to 1}\frac{\ln(x)}{x-1}=\lim_{x\to 1}\frac{\frac{1}{x}}{1}$$
$$=1$$

This cannot be fully checked using a calculator as any attempt to evaluate the expression at $x = 1$ will give an error message. However, plotting the graph and using trace will support our answer:




### Example E.8.3

Evaluate: $\lim_{x\to 0}\frac{\cot(x)}{\ln(x)}$

Let: $f(x)=\cot(x), g(x)=\ln(x)$

We have a $\frac{\infty}{-\infty}$ limit and can use L'Hopital's Rule.

$$\lim_{x\to 0}\frac{\cot(x)}{\ln(x)}=\lim_{x\to 0}\frac{\left(-\frac{1}{\sin^2(x)}\right)}{\left(\frac{1}{x}\right)}$$
$$=-\lim_{x\to 0}\frac{x}{\sin^2(x)}$$

This is $\frac{0}{0}$ and we apply the rule a second time.

$$-\lim_{x\to 0}\frac{x}{\sin^2(x)}=-\lim_{x\to 0}\frac{1}{2\sin(x)\cos(x)}$$
$$=-\lim_{x\to 0}\frac{1}{\sin(2x)}$$
$$=-\infty$$

### Example E.8.4

Evaluate: $\lim_{x\to\infty}\frac{\ln(x)}{x^n}$

We have a $\frac{\infty}{\infty}$ limit and can use L'Hopital's Rule.

$$\lim_{x\to\infty}\frac{\ln(x)}{x^n}=\lim_{x\to\infty}\frac{\frac{1}{x}}{nx^{n-1}}$$
$$=\lim_{x\to\infty}\frac{0}{nx^{n-1}}$$
$$=0$$

**Example E.8.5**

Evaluate: $\lim_{x \to 0} x\ln(x)$

This product is of the form $0 \times -\infty$ and so the expression must be rewritten as:

$$\lim_{x \to 0} x\ln(x) = \lim_{x \to 0} \frac{\ln(x)}{\frac{1}{x}}$$

Next, use the rule: $\lim_{x \to 0} \frac{\ln(x)}{\frac{1}{x}} = \lim_{x \to 0} \frac{\frac{1}{x}}{-\frac{1}{x^2}}$

$$= \lim_{x \to 0} (-x)$$

$$= 0$$

**Example E.8.6**

Evaluate: $\lim_{x \to \infty} x\sin\left(\frac{\pi}{x}\right)$

This product is of the form $0 \times \infty$ and so the expression must be rewritten as:

$$\lim_{x \to \infty} x\sin\left(\frac{\pi}{x}\right) = \lim_{x \to \infty} \frac{\sin\left(\frac{\pi}{x}\right)}{\frac{1}{x}}$$

This is of the form $\frac{0}{0}$ so we can apply L'Hopital's Rule.

$$\lim_{x \to \infty} \frac{\sin\left(\frac{\pi}{x}\right)}{\frac{1}{x}} = \lim_{x \to \infty} \frac{\frac{\pi}{x^2}\cos\left(\frac{\pi}{x}\right)}{-\frac{1}{x^2}}$$

$$= \lim_{x \to \infty} \pi\cos\left(\frac{\pi}{x}\right)$$

$$= \pi$$

**Example E.8.7**

Evaluate: $\lim_{x \to \frac{\pi}{4}} \frac{1-\tan(x)}{\cos(2x)}$

This is of the form $\frac{0}{0}$ so we can apply L'Hopital's Rule.

Let: $f(x) = 1 - \tan(x), f'(x) = -\sec^2(x)$
$g(x) = \cos(2x), g'(x) = -2\sin(2x)$

$$\lim_{x \to \frac{\pi}{4}} \frac{1-\tan(x)}{\cos(2x)} = \lim_{x \to \frac{\pi}{4}} \frac{-\sec^2(x)}{-2\sin(2x)}$$

$$= \lim_{x \to \frac{\pi}{4}} \frac{-\sec^2\left(\frac{\pi}{4}\right)}{-2\sin\left(\frac{\pi}{2}\right)}$$

$$= \frac{\frac{1}{\cos^2\left(\frac{\pi}{4}\right)}}{2\sin\left(\frac{\pi}{2}\right)}$$

$$= \frac{\frac{1}{\left(\frac{1}{\sqrt{2}}\right)^2}}{2}$$

$$= \frac{2}{2}$$

$$= 1$$

## Exercise E.8.1

1. Determine the following limits.

   a $\lim_{x \to 0} \left(\frac{x+\sin 2x}{x-\sin 2x}\right)$

   b $\lim_{x \to \pi} \left(\frac{x-\pi}{\sin 2x}\right)$

   c $\lim_{x \to \frac{\pi}{2}} \left(\frac{\sin 2x}{\cos x}\right)$

2. Determine the following limits.

a $\lim_{x \to \infty} \left( \frac{x}{e^{2x}} \right)$

b $\lim_{x \to \infty} \left( \frac{\ln x}{x} \right)$

c $\lim_{x \to \infty} \left( \frac{2x}{x + \ln x} \right)$

3. Determine the following limits.

a $\lim_{x \to 0} \left( \frac{2x}{x + \sin x} \right)$

b $\lim_{x \to 0} \left( \frac{\cos x - 1}{x^2} \right)$

c $\lim_{x \to 0} \left( \frac{x - \sin x}{x^3} \right)$

4. Determine the following limits.

a $\lim_{x \to \frac{\pi}{2}} \left( \frac{\sin x - 1}{\cos x} \right)$

b $\lim_{x \to 0^+} x \ln\left( 1 + \frac{1}{x} \right)$

c $\lim_{x \to 1} \left( \frac{\ln x - (x-1)}{x - 1} \right)$

5. Determine the following limits, if they exist.

a $\lim_{x \to \frac{\pi}{2}} (\tan x + \sec x)$

b $\lim_{x \to 1} \left( \frac{1}{\ln x} - \frac{1}{x-1} \right)$

c $\lim_{x \to 1} \left( \frac{\ln x}{x^2 - x} \right)$

6. What is wrong in the calculation:

$$\lim_{x \to 0} \left( \frac{\cos x}{x^2} \right) = \lim_{x \to 0} \left( \frac{-\sin x}{2x} \right)$$
$$= \lim_{x \to 0} \left( \frac{-\cos x}{2} \right)$$
$$= -\frac{1}{2}$$

7. Determine the following limits, if they exist.

a $\lim_{x \to \infty} \left( \frac{1}{x} e^x \right)$

b $\lim_{x \to \infty} \left( \frac{x^2}{e^x} \right)$

8. Evaluate the following limits, if they exist.

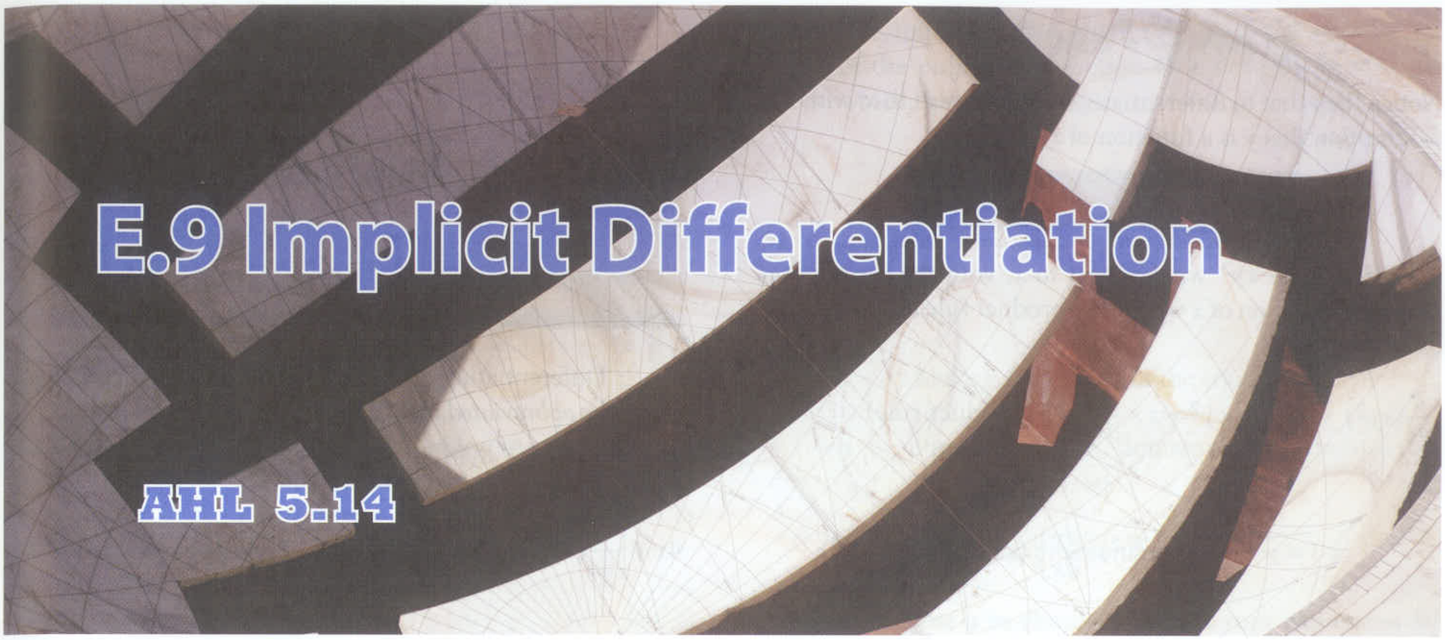
a $\lim_{x \to 0} \frac{x - \sin x}{x^2 e^x}$

b $\lim_{x \to 0} \frac{1 - \cos x}{\sin^2 x}$

c $\lim_{x \to 1} \left( \frac{x^4 - 7x^3 + 8x^2 - 2}{x^3 + 5x - 6} \right)$

Extra questions

Answers



# E.9 Implicit Differentiation

AHL 5.14

## Implicit Differentiation

### Implicit relations

Most of the equations that we have dealt with so far have been expressed in the form $y = f(x)$. For example, $y = \sin(2x)+1$, $y = x^3-2x$, $y = \ln(x-e^x)$, that is, $y$ has been expressed explicitly in terms of $x$ so that for any one given value of $x$ we obtain a unique value of $y$ by substituting the $x$-value into the given equation.

Expressions such as $x^2y+y-2 = 0$, $\sin(xy) = 1$, $e^{x+y} = x+y$, are called implicit equations because these equations define $y$ implicitly as a function of $x$. Note then that although $y = x^2$ defines $y$ as an explicit function of $x$, the equation $y^2+(x+x^2)y+x^3 = 0$ defines $y$ implicitly as functions of $x$ – in fact, we have that two functions are defined implicitly by the equation $y^2+(x+x^2)y+x^3 = 0$ – they are $y = -x$ and $y = -x^2$. We shall see how it is sometimes possible to extract functions from an implicit equation.

It may be possible for an implicit function to be rearranged to form an explicit function. For example, using the equation $x^2y+y-2 = 0$ we have that $(x^2+1)y = 2$ and so, we obtain the equation $y = \dfrac{2}{x^2+1}$ which defines $y$ explicitly in terms of $x$.

Using the implicit function $y^2+(x+x^2)y+x^3 = 0$ we have (after expanding and grouping) that

$$y^2+(x+x^2)y+x^3 = 0 \Leftrightarrow (y+x^2)(y+x) = 0 \Leftrightarrow y = -x$$

or $y = -x^2$. So, we see that in this case two functions are defined implicitly by the equation $y^2+(x+x^2)y+x^3 = 0$.

In fact with more complicated equations it may not be possible to even produce an expression for $y$, i.e. to solve explicitly for $y$. Sometimes even simple equations may not define $y$ uniquely as a function of $x$. For example, if we consider the equation $e^{x+y} = x+y$ we realize that it is not possible to obtain an expression for $y$ explicitly in terms of $x$. The question then arises, "*How can we differentiate equations such as these?*".

We start by considering the equation $x^2y = 2$. As $y$ is implicitly defined as a function of $x$, then, one way of finding the derivative of $y$ with respect to $x$ is to first express $y$ explicitly in terms of $x$:

So, from $x^2y = 2$ we have $y = \dfrac{2}{x^2} \Rightarrow \dfrac{dy}{dx} = -\dfrac{4}{x^3}$.

This method works well, as long as $y$ can be expressed explicitly in terms of $x$.

Now consider the equation $2x^2+y^3-y = 2$. This time it is not possible to express $y$ explicitly in terms of $x$ and so we use a procedure known as implicit differentiation.

The key to understanding how to find $\dfrac{dy}{dx}$ implicitly is to realise that we are *differentiating with respect to x* – so that terms in the equation that involve '$x$'s only can be differentiated as usual but terms that involve '$y$'s must have the chain rule applied to them (and possibly the product rule or quotient rule) because we are assuming that $y$ is a function of $x$.

Before we deal with the equation $2x^2+y^3-y = 2$ we discuss some further examples.

To differentiate $y^3$ with respect to $x$, with the assumption that $y$ is a function of $x$ we use the chain rule as follows: $\dfrac{d}{dx}(y^3) = \dfrac{d}{dy}(y^3)\cdot\dfrac{dy}{dx} = 3y^2\cdot\dfrac{dy}{dx}$

To differentiate $\sin y$ with respect to $x$, with the assumption that $y$ is a function of $x$ we use the chain rule as follows:

$$\frac{d}{dx}(\sin y) = \frac{d}{dy}(\sin y)\cdot\frac{dy}{dx} = \cos y\cdot\frac{dy}{dx}$$

Notice then that to differentiate $y^n$ with respect to $x$, with the assumption that $y$ is a function of $x$ we have:

$$\frac{d}{dx}(y^n) = \frac{d}{dy}(y^n)\cdot\frac{dy}{dx} = ny^{n-1}\cdot\frac{dy}{dx}$$

To differentiate $xy^2$ with respect to $x$, with the assumption that $y$ is a function of $x$ we use the product rule and chain rule as follows:

$$\frac{d}{dx}(xy^2) = \frac{d}{dx}(x)\times y^2 + x\times\frac{d}{dx}(y^2) \text{ (product rule)}$$

$$= 1\times y^2 + x\times\left[\frac{d}{dy}(y^2)\cdot\frac{dy}{dx}\right] \text{ (chain rule for } y^2\text{)}$$

$$= y^2 + x\left[2y\cdot\frac{dy}{dx}\right].$$

And so we have that $\frac{d}{dx}(xy^2) = y^2 + 2xy\cdot\frac{dy}{dx}$.

Now let us return to the equation $2x^2 + y^3 - y = 2$ and find the gradient of the curve at the point (1, 1).

We start by *differentiating both sides of the equation with respect to x*:

i.e. $\frac{d}{dx}(2x^2 + y^3 - y) = \frac{d}{dx}(2)$

Then, we differentiate each term in the expression with respect to $x$:

$$\frac{d}{dx}(2x^2) + \frac{d}{dx}(y^3) - \frac{d}{dx}(y) = 0$$

Use the chain rule $\quad 4x + \frac{d}{dy}(y^3)\cdot\frac{dy}{dx} - \frac{dy}{dx} = 0$

$$4x + (3y^2)\cdot\frac{dy}{dx} - \frac{dy}{dx} = 0$$

Then we group the $\frac{dy}{dx}$ terms and factorise:

$$4x + (3y^2 - 1)\frac{dy}{dx} = 0$$

Then, we solve for $\frac{dy}{dx}$: $\quad \frac{dy}{dx} = -\frac{4x}{3y^2 - 1}$.

The first thing we notice is that the derivative involves both $x$ and $y$ terms. Now sometimes it is possible to simplify so that there are only $x$ terms in the expression and sometimes it can only be left as is. In this case it will be left in terms of $x$ and $y$.

Then, to find the gradient of the curve at the point (1, 1) we substitute the values $x = 1$ and $y = 1$ into the equation of the derivative: $\frac{dy}{dx} = -\frac{4}{3-1} = -2$.

### Example E.9.1

Find the derivatives of these relations.

a $\quad 2x^2 + xy = 5 \qquad$ b $\quad \frac{y}{x} + 3y^2 = 2x^3$

c $\quad x\text{Sin}^{-1}y = e^{2y}$

a Differentiating both sides with respect to $x$ *(which can be abbreviated to diff. b.s.w.r.t x)*:

$$\frac{d}{dx}(2x^2 + xy) = \frac{d}{dx}(5) \quad \therefore \frac{d}{dx}(2x^2) + \frac{d}{dx}(xy) = 0$$

$$4x + \left[\frac{d}{dx}(x)\times y + x\times\frac{d}{dx}(y)\right] = 0 \text{ (Using product rule)}$$

$$\therefore 4x + \left[1\times y + x\frac{dy}{dx}\right] = 0$$

$$\Leftrightarrow x\frac{dy}{dx} = -4x - y$$

$$\Leftrightarrow \frac{dy}{dx} = -\frac{(4x+y)}{x}$$

b Here, the first term must be differentiated using the quotient rule. We consider this term on its own first. Its derivative with respect to $x$ is:

$$\frac{d}{dx}\left(\frac{y}{x}\right) = \frac{x\times\frac{d}{dx}(y) - y\times\frac{d}{dx}(x)}{x^2} = \frac{x\times\frac{dy}{dx} - y}{x^2}$$

Then, *diff b.s.w.r.t.x* we have:

$$\frac{d}{dx}\left(\frac{y}{x} + 3y^2\right) = \frac{d}{dx}(2x^3) \Rightarrow \frac{d}{dx}\left(\frac{y}{x}\right) + \frac{d}{dx}(3y^2) = 6x^2$$

$$\therefore \frac{x\times\frac{dy}{dx} - y}{x^2} + 6y\times\frac{dy}{dx} = 6x^2$$

$$\therefore x\times\frac{dy}{dx} - y + 6x^2y\times\frac{dy}{dx} = 6x^4 \text{ (multiplying through by } x^2\text{)}$$

$$\frac{dy}{dx}(x + 6x^2y) = 6x^4 + y \text{ (grouping the } \frac{dy}{dx} \text{ terms)}$$

$$\therefore \frac{dy}{dx} = \frac{6x^4 + y}{x + 6x^2y}$$

c  *diff b.s.w.r.t.x*: $\frac{d}{dx}(x\mathrm{Sin}^{-1}y) = \frac{d}{dx}(e^{2y})$

(Using product rule for L.H.S and chain rule for R.H.S)

$$\frac{d}{dx}(x) \times \mathrm{Sin}^{-1}y + x \times \frac{d}{dx}(\mathrm{Sin}^{-1}y) = \frac{d}{dy}(e^{2y})\frac{dy}{dx}$$

$$1 \times \mathrm{Sin}^{-1}y + x\frac{d}{dy}(\mathrm{Sin}^{-1}y)\frac{dy}{dx} = 2e^{2y}\frac{dy}{dx} \text{ (chain rule)}$$

$$\mathrm{Sin}^{-1}y + \frac{x}{\sqrt{1-y^2}}\frac{dy}{dx} = 2e^{2y}\frac{dy}{dx}$$

$$\therefore \mathrm{Sin}^{-1}y = \left(2e^{2y} - \frac{x}{\sqrt{1-y^2}}\right)\frac{dy}{dx} \text{ (grouping the } \frac{dy}{dx} \text{ terms)}$$

$$\frac{dy}{dx} = \frac{\sqrt{1-y^2}(\mathrm{Sin}^{-1}y)}{2e^{2y}\sqrt{1-y^2} - x}$$

## Exercise E.9.1

1. Find the first derivative, $\frac{dy}{dx}$, of the following relations in which $y$ depends on $x$.

| | | | |
|---|---|---|---|
| a | $2 + x^2 + y = 6$ | b | $-3 + x^2 + y^2 = 5$ |
| c | $\frac{1}{x^2} + y^2 = 14$ | d | $y + xy = 9$ |
| e | $4 = y + y \times e^x$ | f | $\cos x + xy = 12$ |
| g | $x + \ln(y) = 8$ | h | $\frac{1}{x} + x^3y = -11$ |
| i | $2x + y\sin x = 5$ | j | $(x+y)^2 = 12$ |
| k | $x^4 = y + y^3$ | l | $2\sqrt{x+y} = x$ |

2. $f(x)$ is a relation on a real variable $x$ such that $e^{f(x)} - f(x) = e^5 - 5$. Find the coordinates of the point for which $x = 1$ and the gradient of the graph of the function at this point.

3. A curve has equation $\frac{e^{x^2y}}{x} + 2x = 3$.

Differentiate the equation implicitly and hence prove that:

$$\frac{e^{x^2y}}{x} = 2x + e^{x^2y}\left(2xy + x^2\frac{dy}{dx}\right).$$

4. Use implicit differentiation to find the coordinates of the points on the circle $x^2 - 3x + y^2 - 4y = 7$ for which the gradient is 2.

5. Consider the conic section with equation:

$x^2 + xy - y^2 = 20$.

a Make $y$ the subject of the equation.

b Prove that the domain of the relation is $]-\infty, 4] \cup [4, \infty[$.

c Find an expression for $\frac{dy}{dx}$.

d Use a to eliminate $y$ from your expression for $\frac{dy}{dx}$.

e Hence prove that as $x \to \pm\infty$, $\frac{dy}{dx} \to \frac{5 \pm \sqrt{5}}{2\sqrt{5}}$.

f What type of curve is represented by $x^2 + xy - y^2 = 20$?

6. A curve has equation $x^4 + y^4 = 16$.

a Find the domain and range of the relation.

b Express the gradient, $\frac{dy}{dx}$, in terms of $x$ and $y$.

c Eliminate $y$ from your expression in part b.

d What is the gradient in the region of the $y$-axis?

Consider the family of relations $x^{2n} + y^{2n} = k^{2n}$ where $k$ is a constant and $n$ is a positive integer.

e Find the domain and range of the relation.

f Express the gradient, $\frac{dy}{dx}$, in terms of $x$ and $y$ and hence describe the form of the graph of the relation as $n$ becomes large.

7.

a If $pv^{\Upsilon} = c$ where c and $\Upsilon$ are real constants, find $\frac{dv}{dp}$.

b Find $\frac{dy}{dx}$ if $\frac{x^m}{y^n} = \frac{m}{n}xy$.

8. Find the slope of the curve

a $x^3 + y^3 - x^2y = 7$ at $(1, 2)$

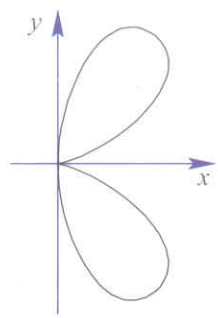
b $x^3 + y^3 - 3kxy = 0$ at $\left(\frac{3}{2}k, \frac{3}{2}k\right)$.

9. Find $\frac{dy}{dx}$ if:

a $\log_e(xy) = y, x > 0$

b $x\text{Tan}^{-1}(y) = x + y$.

10. The graph of the curve $(x^2 + y^2)^2 = 4xy^2$ is shown alongside.




a Find the gradient of the curve at the point where $x = 1$. Explain your result.

b Find the gradients of the curve where $y = ½$, giving your answers to 2 decimal places.

## Related Rates

So far we have only dealt with rates of change that involve one independent variable. For example, the volume, $V$ units$^3$, of a sphere of radius $r$ units is given by $V = \frac{4}{3}\pi r^3$. To find the rate of change of the volume with respect to its radius we differentiate with respect to $r$:

i.e. $\frac{dV}{dr} = \frac{4}{3}\pi \times 3r^2 = 4\pi r^2$.

Now consider this sphere being placed in an acid solution so that it dissolves in such a way that:

1. it maintains its spherical shape, and

2. its radius is decreasing at a rate of 1 cm/hr.

How can we find the rate at which its volume is changing when the sphere's radius is 2 cm?

Note that we are looking for the *rate of change of volume*, that is, we want to find $\frac{dV}{dt}$ (not $\frac{dV}{dr}$ as we found previously – when we specifically requested the rate of change with respect to $r$). The difference here is that we want the rate of change of one quantity (in this case the volume) which is related to a second variable (in this case the radius $r$) which is itself changing.

Problems of this type are known as related rates problems and are usually solved by making use of the chain rule.

We now consider the problem at hand. We have:

**Want:** *rate of change of volume* that is, we want to find $\frac{dV}{dt}$.

**When:** $r = 2$.

**Given:** radius is decreasing at a rate of 1 cm/hr, $\frac{dr}{dt} = -1$

**Need:** This is the tricky bit. Knowing that we will need to use the chain rule, we start by writing down the chain rule with the information we have. Then we try to fill in the missing pieces.

This will often lead to what we need.

Step 1: $\frac{dV}{dt} = \left[\frac{\ }{\ }\right] \times \frac{dr}{dt}$

Step 2: Ask yourself the following question:

"*What do I need in the missing space to complete the chain rule?*"

The missing piece of information in this case is $\frac{dV}{dr}$.

That is, we have $\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$, which works.

Step 3: Once you have decided on what you need, then find an equation that will enable you to differentiate.

Some warning! Step 3 is the tough bit in the question. Sometimes we are lucky and we know of an equation but sometimes we need to somehow 'create' the equation.

In this case we do have an equation; $V = \frac{4}{3}\pi r^3 \therefore \frac{dV}{dr} = 4\pi r^2$.

And so, using the chain rule we have $\frac{dV}{dt} = 4\pi r^2 \times \frac{dr}{dt}$.

Note: It is **very important** not to substitute any values until the very end.

The last step is to find $\frac{dV}{dt}$ at the specified radius with the given rate, $\frac{dr}{dt} = -1$.

That is, $\frac{dV}{dt} = 4\pi(2)^2 \times -1 = -16\pi$.

So, the volume is decreasing at $16\pi$ cm$^3$/hr.

### Example E.9.2

The radius of a circular oil patch is increasing at a rate of 1.2 cm per minute. Find the rate at which the surface area of the patch is increasing when the radius is 25 cm.

From the data, $\frac{dr}{dt} = 1.2$.

This is the mathematical formulation of the statement 'the radius of a circular oil patch is increasing at a rate of 1.2 cm per minute' where $r$ is the radius and $t$ is the time (in the units given in the question). The radius is increasing and so the rate is positive. The next step is to identify the rate of change that we have been asked to calculate. In this case, the question asks: 'find the rate at which the surface area of the patch is increasing'.

If we define the area as $A$ cm², the required rate is $\frac{dA}{dt}$.

So we have: **Want:** $\frac{dA}{dt}$

**When:** $r = 25$

**Given:** $\frac{dr}{dt} = 1.2$

**Need:** (chain rule): $\frac{dA}{dt} = \left[\,-\,\right] \times \frac{dr}{dt}$.

The missing piece must therefore be $\frac{dA}{dr}$.

Therefore, we have, $\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$.

We need to find an expression for $A$ in terms of $r$.

This can be done by looking at the geometry of the situation. The oil patch is circular and so the area is given by $A = \pi r^2 \Rightarrow \frac{dA}{dr} = 2\pi r$.

Substituting into the chain rule gives: $\frac{dA}{dt} = 2\pi r \times \frac{dr}{dt}$.

Then, with $r = 25$ and $\frac{dr}{dt} = 1.2$ we have:

$$\frac{dA}{dt} = 2\pi(25) \times 1.2 = 60\pi \approx 188.5 \text{ cm}^2\text{min}^{-1}.$$

That is, the area is increasing at approximately 188.5 cm²min⁻¹.

Note: A useful check that the chain rule has been used appropriately is to use the units of the quantities involved. For Example 6.3.14 we have that:

$$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt} = \text{cm}^2\text{cm}^{-1} \times \text{cm}^1\text{min}^{-1} = \text{cm}^2\text{min}^{-1}$$

which is the correct unit for $\frac{dA}{dt}$.

### Example E.9.3

The volume of a cube is increasing at 24 cm³s⁻¹. At what rate are the side lengths increasing when the volume is 1,000 cm³?

We start by determining what variables are involved and see if a diagram might be helpful – usually one is (even if it's only used to visualize the situation). In this case we are talking about a volume and a length, so we let $V$ cm³ denote the volume of the cube of side length $x$ cm, giving us the expression $V = x^3$.

Next we list all of the information according to our *want*, *when*, *given* and *need*:

**Want:** $\frac{dx}{dt}$

**When:** $V = 1{,}000$

**Given:** $\frac{dV}{dt} = 24$

**Need:** (chain rule) $\frac{dx}{dt} = \left[\,-\,\right] \times \frac{dV}{dt}$ we need $\frac{dx}{dV}$.

So that $\frac{dx}{dt} = \frac{dx}{dV} \times \frac{dV}{dt}$.

However, we have $V$ as a function of $x$ and so it will be easier to first find $\frac{dV}{dx}$ and then use the fact that:

$$\frac{dx}{dV} = \frac{1}{\frac{dV}{dx}}.$$

Then, as $V = x^3 \Rightarrow \frac{dV}{dx} = 3x^2 \Rightarrow \frac{dx}{dt} = \frac{1}{3x^2} \times \frac{dV}{dt}$,

We know $\frac{dV}{dt} = 24$ but, still need a value for $x$.

From $V = x^3$ we have $1000 = x^3 \therefore x = 10$.

So, $\frac{dx}{dt} = \frac{1}{3(10)^2} \times 24 = \frac{8}{100} = 0.08$.

That is, the side lengths are increasing at 0.08 cms⁻¹.

It is important to realize that when we reach the '**Need:**' stage there are more ways than one to use the chain rule.

For example, with Example E.9.3, rather than using $\frac{dx}{dt} = \frac{dx}{dV} \times \frac{dV}{dt}$ and then realizing that we need to find $\frac{dV}{dx}$ and then invert it, we could have used the chain rule as follows:

$$\frac{dV}{dt} = \frac{dV}{dx} \times \frac{dx}{dt} \text{ so that } 24 = 3x^2 \times \frac{dx}{dt} \Leftrightarrow \frac{dx}{dt} = \frac{24}{3x^2}.$$

Using the chain rule in this manner has a certain 'logical flow' to it, in that everything seems to 'fit nicely'. But remember, as long as the chain rule expression contains the '*need*', '*want*' and '*given*' it should not make much difference at the end. All that we can say is that as you solve more and more of these problems you will be able to make the 'best' decision available at the time.

### Example E.9.4

A container in the shape of an inverted right circular cone of base radius 10 cm and height 50 cm has water poured into it at a rate of 5 cm³min⁻¹. Find the rate at which the level of the water is rising when it reaches a height of 10 cm.

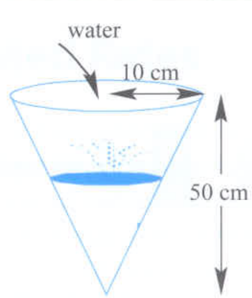


Figure A

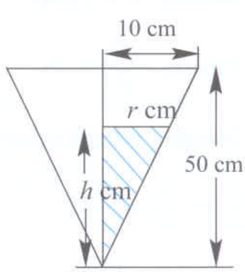


Figure B

Let the water level at time $t$ min have a height $h$ cm with a corresponding radius $r$ cm and volume $V$ cm³.

We now list our requirements:

**Want:** $\frac{dh}{dt}$

**When:** $h = 10$

**Given:** $\frac{dV}{dt} = 5$

**Need:** (chain rule) $\frac{dV}{dt} = \left[\frac{}{}\right] \times \frac{dh}{dt}$ we need $\frac{dV}{dh}$.

Before we can find $\frac{dV}{dh}$ we will need to find an expression for $V$ in terms of $h$. We do this by making use of Figure B – a cross-section of the inverted cone. The information in Figure B prompts us to make use of similar triangles.

We then have, $\frac{50}{10} = \frac{h}{r} \Leftrightarrow r = \frac{1}{5}h$.

The volume of water in the cone when it reaches a height $h$ cm is given by: $V = \frac{1}{3}\pi r^2 h$.

Then, substituting the expression $r = \frac{1}{5}h$ into the volume equation we have $V = \frac{1}{3}\pi\left(\frac{1}{5}h\right)^2 h = \frac{\pi}{75}h^3 \Rightarrow \frac{dV}{dh} = \frac{\pi}{25}h^2$.

We can now complete the chain rule:

$$\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt} \therefore \frac{dV}{dt} = \frac{\pi}{25}h^2 \times \frac{dh}{dt}$$

$$5 = \frac{\pi}{25}h^2 \times \frac{dh}{dt} \quad \Leftrightarrow \frac{dh}{dt} = \frac{125}{\pi h^2}$$

Then, when $h = 10$, we have $\frac{dh}{dt} = \frac{125}{100\pi} \approx 0.3978$, i.e. approximately 0.4 cms⁻¹.

### Exercise E.9.2

1. The radius of a circle is increasing at 2 cm/s. Find the rate at which:

   a - its area is increasing and b - its circumference is increasing.

2. The side lengths of a square are increasing at a rate of 3 cm/s. Find the rate at which the area of the square is increasing when its side length is 1 cm.

3. The sides of an equilateral triangle are decreasing at a rate of $\sqrt{6}$ cm/s. Find the rate of change of:

   a - the area of the triangle and b - the altitude of the triangle.

4. A solid 400 g metal cube of side length 10 cm expands uniformly when heated. If the length of its sides expand at 0.5 cm/hr, find the rate at which, after 5 hours:

   a its volume is increasing.

   b its surface area is increasing.

   c its density is changing.

5. A drinking glass is shaped in such a way that the volume of water in the glass when it reaches a height $h$ cm is given by $V = \frac{1}{5}h^3$ cm$^3$.

Water is poured into the glass at 2 cm$^3$s$^{-1}$. At what rate is the water level rising when the depth of water is 3 cm?

6. An ice cube, while retaining its shape, is melting and its side lengths are decreasing at 0.02 cm/min. Find the rate at which the volume is changing when the sides are 2 cm.

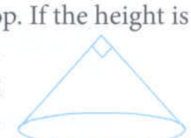
7. A liquid is pumped into an upright cylindrical tank of radius 1.5 m at a rate of 0.25 m$^3$s$^{-1}$.

At what rate is the depth of the liquid increasing when it reaches:

a a depth of 1.25 m?

b a volume of $10\pi$ m$^3$?

8^: A conical pile of sand with a constant vertical angle of 90° is having sand poured onto the top. If the height is increasing at the rate of 0.5 cm min$^{-1}$, find the rate at which sand is being poured when the height is 4 cm, giving an exact answer.



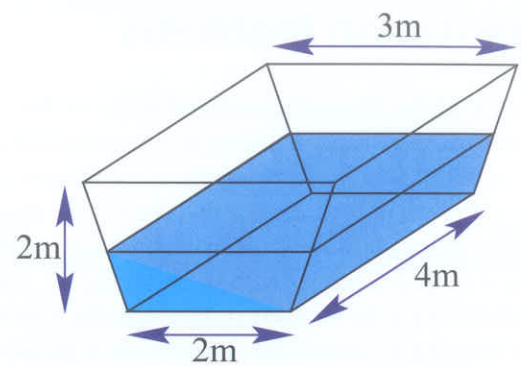
9. An aeroplane flies over an airport at an altitude of 10,000 metres and at a speed of 900 kmh$^{-1}$. Find the rate at which the actual distance from the airport is increasing 2 minutes after the aeroplane was directly over the airport, correct to the nearest whole number.

10. The temperature inside a chemical reaction vessel, initially 35°C is rising at 7°C per hour.

The rate at which the reaction happens is modelled by the function: rate $= \frac{t}{12} + 3$, where $t$ is the temperature of the reaction vessel in °C. Find the rate at which the reaction is occurring after 5 hours.

11. A racing car, travelling at 180 km per hour, is passing a television camera on a straight road. The camera is 25 metres from the road. If the camera operator follows the car, find the rate (in radians per second) at which the camera must pan (rotate) at the moment when the car is at its closest to the camera.

12. The diagram shows a water trough. Water is being poured into this trough at 2.4 cubic metres per minute.




a Find an expression for the volume of water in the trough in terms of its depth.

b Find the rate at which the water level is rising when the depth is 0.5 metres.

c Find the rate at which the exposed surface area of the water is increasing after 1 minute.

13. A square-based pyramid with a fixed height of 20 metres is increasing in volume at 2 m$^3$min$^{-1}$. Find the rate at which the side length of the base is increasing when the base has an area of 10 m$^2$. Give an exact answer with a rational denominator.

14. The length of the edge of a regular tetrahedron is increasing at 2.5 cms$^{-1}$. Find the rate at which the volume is increasing when the edge is 4 cm.

15. A man 1.8 m tall is walking directly away from a street lamp 3.2 m above the ground at a speed of 0.7 m/s. How fast is the length of his shadow increasing?

16. A ladder 10 m long rests against a vertical wall. The bottom of the ladder, while maintaining contact with the ground, is being pulled away from the wall at 0.8 m/s. How fast is the top of the ladder sliding down the wall, when it is 2 m from the ground?

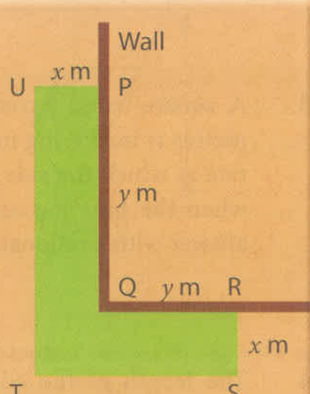
Extra questions

## Optimization Problems

Chapter E4 of the SL text introduced some of the calculus techniques that are useful in finding the maxima and minima of graphs. We now look at using these techniques to solve problems. Problems that require the use of this theory can be found in many real-life situations: manufacturers wanting to minimize their costs, designers wanting to maximize the available space to be used (under specific constraints), farmers wanting to maximize the area of a paddock at a minimum cost, etc. These types of problems often require the construction of an appropriate function that models a particular situation, from which some optimum quantity can be derived or a critical value found for which this optimum quantity exists. We now consider a number of examples to highlight how differential calculus can be used to solve such problems.

### Example E.9.5

The points PQR form the corner of a house, where angle PQR is a right-angle. Running parallel to these walls is a garden patch. There is only 20 metres of fencing available to create the enclosure PUTSRQ, where PU = RS = $x$ and PQ = QR = $y$.




a Express ST in terms of $x$ and $y$.

b Find an expression for $y$ in terms of $x$.

c What area does this garden patch cover (give your answer in terms of $x$)?

d Find the maximum area enclosed by this fence and the walls. Justify your answer.

a ST = UP + QR = $x + y$.

b There is 20 m of fencing available, therefore, PU + UT + TS + SR = 20

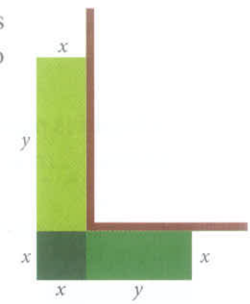
That is, $2x + 2(x + y) = 20$

and so $y = 10 - 2x$ – Eq. (1).

Note: As $y \geq 0 \Rightarrow 10 - 2x \geq 0 \Leftrightarrow x \leq 5$. We must also have that $x \geq 0$.

That is, there is a restriction on $x$, namely $0 \leq x \leq 5$.

c The required area, $A$ m², is found by breaking the area into three sections, so that:




$A = xy + xy + x^2$

$= 2xy + x^2$

$= 2x(10 - 2x) + x^2$, given that $y = 10 - 2x$

$= 20x - 3x^2, 0 \leq x \leq 5$

d To find stationary points we first determine the first derivative:

$$\frac{dA}{dx} = 20 - 6x$$

and then equate this to zero. $20 - 6x = 0 \Rightarrow x = \frac{20}{6} = \frac{10}{3}$

To establish that this is a maximum, we can:

i Note that the graph is a 'vertex up' parabola.

ii To the left of the turning point the derivative is –ve and to the right it is +ve.

iii The second derivative (–6) is negative.

All three tests indicate that we have a maximum point. We also must observe that it lies within the domain.

$$A_{max} = 20\left(\frac{10}{3}\right) - 3\left(\frac{10}{3}\right)^2 = \frac{100}{3}\text{ m}^2$$

### Example E.9.6

In the lead-up to the Christmas shopping period, a toy distributor has produced the following cost and revenue models for one of his toys:

Cost: $C(x)=2.515x-0.00015x^2, 0\le x\le 6500$

Revenue: $R(x)=7.390x-0.0009x^2, 0\le x\le 6500$

where $x$ is the number of units produced.

What is the maximum profit that the distributor can hope for using these models?

The profit is found by determining the Revenue – Cost, so, letting \$ $P(x)$ denote the profit made for producing $x$ units, we have:

$$\begin{aligned}P(x)&=R(x)-C(x)\\&=\left(7.390x-0.0009x^2\right)-\left(2.515x-0.00015x^2\right)\\&=4.875x-0.00075x^2\end{aligned}$$

Next, using the derivative to find any stationary points:

$P'(x)=4.875-0.0015x$ and equating to zero:

$$\begin{aligned}4.875-0.0015x&=0\\x&=\frac{4.875}{0.0015}\\&=3250\end{aligned}$$

The discrimination tests all indicate that this is a maximum and so:

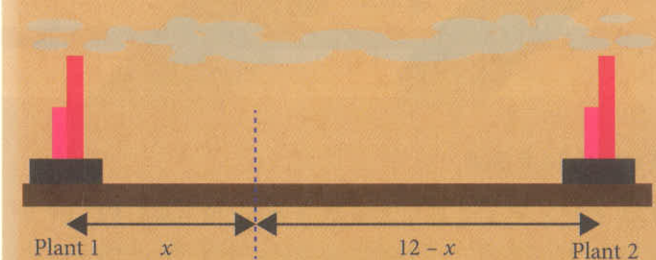
$$\begin{aligned}P_{max}(x)&=4.875(3250)-0.00075(3250)^2\\&=7921.875\end{aligned}$$

The maximum profit the distributor will make is achieved by making 3250 units and is \$7921.875 ≈ \$7 922.

### Example E.9.7

Two heavy industrial plants are located 12 kilometres apart. It is found that the concentration of particulate matter in parts per million in the pollution created at a plant varies as the reciprocal of the square of the distance from the source. If plant 1 emits eight times the particulate matter of plant 2, the combined concentration, C, of particulate matter at any point between the two plants is found to be modelled by:

$$C(x)=\frac{8}{x^2}+\frac{1}{(12-x)^2}, 0.5\le x\le 11.5$$




What is the minimum concentration of particulate matter that there can be between the two plants? How far from Plant 1 will this occur?

We need to determine where the stationary points occur, that is, we solve for $C'(x)=0$.

$$\begin{aligned}C(x)&=\frac{8}{x^2}+\frac{1}{(12-x)^2}\\&=8x^{-2}+(12-x)^{-2}\\C'(x)&=-16x^{-3}+2(12-x)^{-3}\end{aligned}$$

Equating to zero:

$$\begin{aligned}-16x^{-3}+2(12-x)^{-3}&=0\\-\frac{16}{x^3}+\frac{2}{(12-x)^3}&=0\\\frac{16}{x^3}&=\frac{2}{(12-x)^3}\\\frac{x^3}{(12-x)^3}&=8\\\frac{x}{(12-x)}&=2\\x&=2(12-x)\\x&=24-2x\\3x&=24\\x&=8\end{aligned}$$

Testing for the nature of this point:

$$C'(7)=-\frac{16}{343}+\frac{2}{125}\approx-0.03 \searrow$$

$$C'(9)=-\frac{16}{729}+\frac{2}{27}\approx 0.05 \nearrow$$

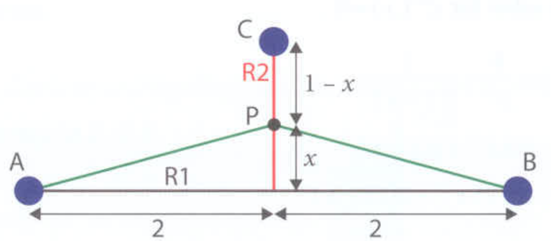
so the point is a minimum.

$$C(8)=\frac{8}{8^2}+\frac{1}{(12-8)^2}$$
$$=\frac{3}{16}$$

So the minimum concentration is $\frac{3}{16}$ parts per million and occurs 8 km from Plant 1.

### Example E.9.8

Two shops, A and B linked by a straight road, R1, are 4 kilometres apart. A second straight road, R2, bisects R1 at right angles. One kilometre along R2 from where it bisects R1 a third shop, C, can be found. A bus stop is to be placed on R2, somewhere between shop C and R1. Where should the bus stop be placed so that the sum of the direct distances from the shops to the bus stop is a minimum?




Let P be the location of the bus stop and $x$ km the distance from R1 to P along R2. This means that $x \geq 0$ ($x = 0$ if on R1) but $x \leq 1$ ($x = 1$ if at C).

Let the sum of the distances be S km, then, S = AP + PC + BP

Using Pythagoras' Theorem we have:

$AP^2 = 2^2 + x^2 \Rightarrow AP=\sqrt{4+x^2}$, as $AP \geq 0$ and,

$BP^2 = 2^2 + x^2 \Rightarrow BP=\sqrt{4+x^2}$, as $BP \geq 0$

Therefore: $S = 2\sqrt{4+x^2}+(1-x), 0 \leq x \leq 1$

Next, we find the derivative and equate to zero.

$$\frac{dS}{dx}=2\times\frac{1}{2}\times 2x\times\frac{1}{\sqrt{4+x^2}}+(-1)$$
$$=\frac{2x}{\sqrt{4+x^2}}-1$$
$$=\frac{2x-\sqrt{4+x^2}}{\sqrt{4+x^2}}$$

Letting $\frac{dS}{dx}=0$:

$$\frac{2x-\sqrt{4+x^2}}{\sqrt{4+x^2}}=0$$
$$2x-\sqrt{4+x^2}=0$$
$$2x=\sqrt{4+x^2}$$
$$4x^2=4+x^2$$
$$3x^2=4$$
$$x=\pm\sqrt{\frac{4}{3}}$$
$$\approx\pm 1.1547$$

However, neither one of these critical values lies in the domain $S$ ( = [0, 1]). This means that we will need to look at an end-point minimum.

When $x = 0$, $S = 2\sqrt{4}+1=5$, giving us a maximum.

When $x = 1$, $S = 2\sqrt{4+1^2}+(1-1)=2\sqrt{5}$

Hence, the minimum value of $S$ occurs at $x = 1$ and has a value of $2\sqrt{5}$ km (approx. 4.47 km). Therefore, the bus stop should be placed at shop C.

### Exercise E.9.3

1. A ball is thrown upwards and after $t$ seconds reaches a height of $h$ m above the ground. The height of the ball at time $t$ is given by the equation $h = 19.6t - 4.9t^2 + 3$. What is the maximum height that the ball will reach from the ground?

2. The running cost, \$$C$ per kilometre for a new car is modelled by the equation $C = 20 + 0.2v^2 - 0.6v$, where $v$ km/h is the average speed of the car during a trip.

   a At what speed should the car be driven to minimize the running cost per kilometre?

b What is the minimum running cost per km for this car?

c Comment on your answers.

3. The total revenue, $\$R$, that a company can expect after selling $x$ units of its product – GIZMO – can be determined by the equation:

$R = -x^3 + 510x^2 + 72000x, x \geq 0.$

a How many units should the company produce to maximize their revenue?

b What is the maximum revenue to be made from the sales of GIZMOs?

4. A retailer has determined that the monthly costs, $\$C$, involved for ordering and storing $x$ units of a product can be modelled by the function:

$$C(x) = 2.5x + \frac{7500}{x}, 0 < x \leq 250$$

What is the minimum monthly cost that the retailer can expect? Note that $x$ is an integer value.

5. The marketing department at DIBI Ltd. have determined that the demand, at $\$d$ per unit, for a product can be modelled by the equation:

$$d = \frac{80}{\sqrt{x}}$$

where $x$ is the number of units produced and sold. The total cost, $\$C$, of producing $x$ items given by:

$C = 200 + 0.2x$

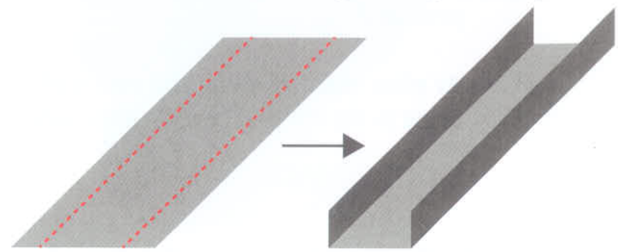
What price will yield a maximum profit?

6. The cross-section of a small hill is modelled by the curve with equation

$$y = \frac{1}{8}x^2 \sin\left(\frac{1}{2}x\right), 0 \leq x \leq 2\pi$$

where $x$ metres is the horizontal distance from the point O and $y$ metres is the corresponding height. What is the maximum height of the hill?

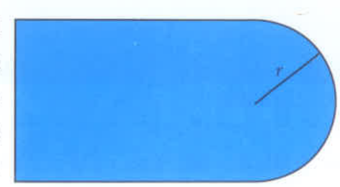
7. A 10 metres long sheet of tin of width 60 centimetres is to be bent to form an open gutter having a rectangular cross-section. Find the maximum volume of water that this 10 metres stretch of guttering can carry.



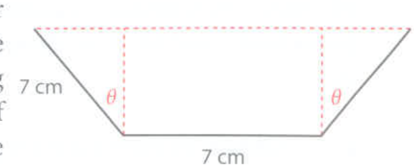
8. A 20-metre long piece of wire is bent into a rectangular shape. Find the dimensions of the rectangle that encloses the maximum area.

9. If $x + y = 8$, find the minimum value of $N = x^3 + y^3$.

10. A swimming pool is constructed as a rectangle and a semicircle of radius $r$ m. The perimeter of the pool is to be 50 metres. Find the value of $r$ and the dimensions of the rectangular section of the pool if the surface area of the pool is to be a maximum.




11. A roof gutter is to be made from a long flat sheet of tin 21 cm wide by turning up sides of 7 cm so that it has a trapezoidal cross-section as shown in the diagram. Find the value of that will maximize the carrying capacity of the gutter.



12. At the Happy Place amusement park, there is a roller coaster ride named 'The Not-So-Happy Ride'. A section of this ride has been created using a scaled version of the model given by the equation:

$$y = \sin x + \frac{1}{2}\sin 2x, 0 \le x \le 2\pi$$

a Sketch the graph of this curve.

b What is the maximum drop that this ride provides?

c At what point(s) along the ride will a person come to the steepest part(s) of the ride?

13. A rectangle is cut from a circular disc of radius 18 metres. Find the maximum area of the rectangle.

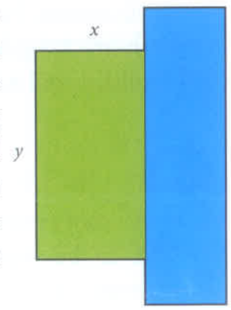
14. Two real numbers $x$ and $y$ are such that $x + y = 21$. Find the value of $x$ for which:

a the product $xy$ is a maximum.

b the product $xy^3$ is a maximum.

15. If $x + y = 12$, find the minimum value that $x^2 + y^2$ can have.

16. A farmer wishes to fence off a rectangular paddock using an existing stretch of a river as one side. The total length of fencing available is 100 m. Let $x$ m and $y$ m denote the width and length of this rectangular paddock respectively, and let $A$ m$^2$ denote its area.

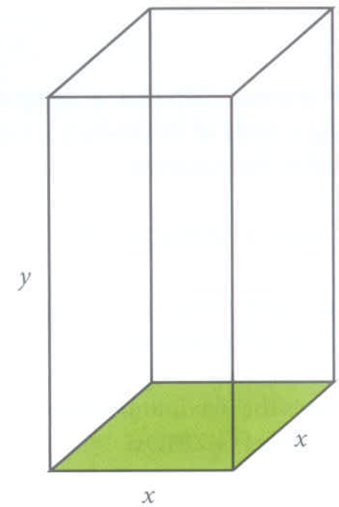



a Obtain an expression for $y$ in terms of $x$.

b Find an expression for $A$ in terms of $x$, stating any restrictions on $x$.

c Determine the dimensions which will maximize the area of the paddock.

17. A closed rectangular box with square ends is to be constructed in such a way that its total surface area is 400 cm$^2$. Let $x$ cm be the side length of the ends and $y$ cm its height.

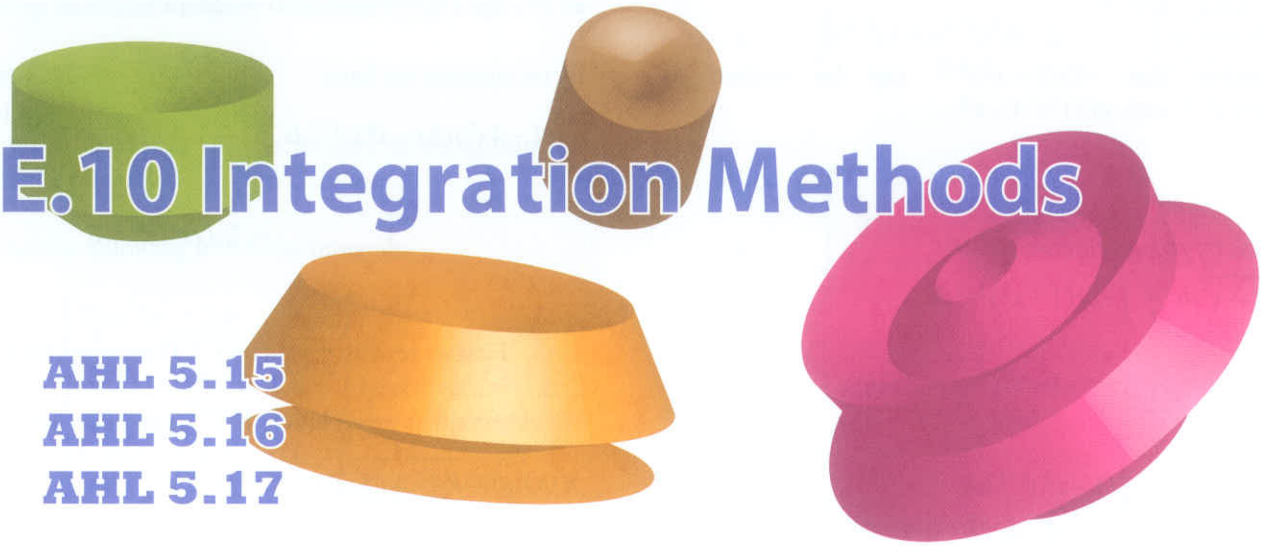



a Obtain an expression for y in terms of x, stating any restrictions on x.

b Find the largest possible volume of all such boxes.

Extra questions

Answers



# E.10 Integration Methods

**AHL 5.15**
**AHL 5.16**
**AHL 5.17**

## Further Integration

We can obtain the antiderivative, $F(x) + c$, of a function $f(x)$ based on the result that $\frac{d}{dx}(F(x)) = f(x)$.

That is, If $\frac{d}{dx}(F(x)) = f(x)$ then $\int f(x)dx = F(x) + c$

For example, if we know that $\frac{d}{dx}(\sin 5x) = 5\cos 5x$, then:

$\int 5\cos 5x\, dx = \sin 5x + c$.

Similarly, if $\frac{d}{dx}\left(\ln\left(x^2+1\right)\right) = \frac{2x}{x^2+1}$, then:

$\int \frac{2x}{x^2+1}dx = \ln\left(x^2+1\right) + c$.

We are using recognition to obtain antiderivatives. Such a skill is crucial to becoming successful at finding more complex antiderivatives.

One particularly important result is based on the chain rule, from which we obtained the generalised 'power rule' for differentiation;

$$\frac{d}{dx}\left(\left[f(x)\right]^n\right) = n.f'(x)\left[f(x)\right]^{n-1}.$$

From this result we have:

$$\int \frac{d}{dx}([f(x)]^n)dx = \int nf'(x)[f(x)]^{n-1}dx$$

so that $\int nf'(x)[f(x)]^{n-1}dx = [f(x)]^n + c$

This leads to the result:

$$\int g'(x)[g(x)]^n dx = \frac{1}{n+1}[g(x)]^{n+1} + c$$

The use of this result is dependent on an ability to recognise the expressions $g(x)$ and its derivative $g'(x)$ within the integrand. We consider a number of examples.

**Example E.10.1**

Find the indefinite integral of the following.

a $2x(x^2+9)^5$ b $(3x^2+1)(x^3+x)^2$

c $-2x\sqrt{1-x^2}$

a We observe that $2x(x^2+9)^5$ can be written as $g'(x)[g(x)]^5$ with $g(x) = x^2+9$.

Therefore, by recognition we have:

$$\int 2x(x^2+9)^5 dx = \frac{1}{5+1}(x^2+9)^{5+1} + c$$
$$= \frac{1}{6}(x^2+9)^6 + c$$

b We observe that $(3x^2+1)(x^3+x)^2$ can be written as $g'(x)[g(x)]^2$ with $g(x) = x^3+x$.

Therefore, by recognition we have:

$$\int (3x^2+1)(x^3+x)^2 dx = \frac{1}{2+1}(x^3+x)^{2+1} + c$$
$$= \frac{1}{3}(x^3+x)^3 + c$$

c We first express $-2x\sqrt{1-x^2}$ in the power form, $-2x(1-x^2)^{1/2}$.

We observe that $-2x(1-x^2)^{1/2}$ can be written as $g'(x)[g(x)]^{1/2}$ with $g(x) = 1-x^2$.

Therefore, by recognition we have:

$$\int -2x\sqrt{1-x^2}dx = \int -2x(1-x^2)^{1/2}dx$$

$$= \frac{1}{\frac{1}{2}+1}(1-x^2)^{\frac{1}{2}+1}$$

$$= \frac{2}{3}(1-x^2)^{3/2}+c$$

$$= \frac{2}{3}\sqrt{(1-x^2)^3}+c$$

**Example E.10.2**

Find the indefinite integral of the following.

a $\dfrac{3x^2}{(x^3+4)^4}$ b $\dfrac{2-4x^3}{\sqrt{2x-x^4}}$

c $\dfrac{1}{x+1}\sqrt{\ln(x+1)}$.

a We rewrite $\dfrac{3x^2}{(x^3+4)^4}$ as $3x^2(x^3+4)^{-4}$.

We observe that $3x^2(x^3+4)^{-4}$ can be written as $g'(x)[g(x)]^{-4}$ with $g(x) = x^3+4$.

Therefore, by recognition we have:

$$\int 3x^2(x^3+4)^{-4}dx = \frac{1}{-4+1}(x^3+4)^{-4+1}+c$$

$$= -\frac{1}{3}(x^3+4)^{-3}+c$$

$$= -\frac{1}{3(x^3+4)^3}+c$$

b First we rewrite $\dfrac{2-4x^3}{\sqrt{2x-x^4}}$ as $(2-4x^3)(2x-x^4)^{-1/2}$.

Then, we observe that $(2-4x^3)(2x-x^4)^{-1/2}$ can be written as $g'(x)[g(x)]^{-1/2}$ with $g(x) = 2x-x^4$.

By recognition we have:

$$\int(2-4x^3)(2x-x^4)^{-1/2}dx = \frac{1}{-\frac{1}{2}+1}(2x-x^4)^{-\frac{1}{2}+1}+c$$

$$= 2\sqrt{2x-x^4}+c$$

c First we rewrite $\dfrac{1}{x+1}\sqrt{\ln(x+1)}$ as $\dfrac{1}{x+1}[\ln(x+1)]^{1/2}$

We observe that $\dfrac{1}{x+1}[\ln(x+1)]^{1/2}$ can be written as:

$g'(x)[g(x)]^{1/2}$ with $g(x) = \ln(x+1)$.

By recognition we have:

$$\int\frac{1}{x+1}[\ln(x+1)]^{1/2}dx = \frac{1}{\frac{1}{2}+1}[\ln(x+1)]^{\frac{1}{2}+1}+c$$

$$= \frac{2}{3}[\ln(x+1)]^{3/2}+c$$

$$= \frac{2}{3}\sqrt{[\ln(x+1)]^3}+c$$

What happens if the expression is not exactly in the form:

$\int g'(x)[g(x)]^n dx$, but only differs by some multiple?

That is, what happens when we have $\int x(x^2+3)^4dx$ or $\int 5x(x^2+3)^4dx$ rather than $\int 2x(x^2+3)^4dx$?

As the expressions only differ by a multiple, we manipulate them so that they transform into $\int g'(x)[g(x)]^n dx$. For example:

$$\int x(x^2+3)^4dx =$$

$$\frac{1}{2}\int 2x(x^2+3)^4dx = \frac{1}{2}\times\frac{1}{5}(x^2+3)^5+c = \frac{1}{10}(x^2+3)^5+c$$

(i.e. multiply and divide by 2.)

$$\int 5x(x^2+3)^4dx =$$

$$5\int x(x^2+3)^4dx = \frac{5}{2}\int 2x(x^2+3)^4dx = \frac{5}{2}\times\frac{1}{5}(x^2+3)^5+c$$

(i.e. 'take' 5 outside the integral sign, then multiply and divide by 2.)

$$= \frac{1}{2}(x^2+3)^5+c$$

These manipulation skills are essential for successfully determining indefinite integrals by recognition.

### Exercise E.10.1

For this set of exercises, use the method of recognition to determine the integrals.

1. Find the following indefinite integrals.

a $\int 10x\sqrt{5x^2+2}\,dx$ b $\int \frac{x^2}{(x^3+4)^2}dx$

c $\int -6x(1-2x^2)^3 dx$ d $\int 3\sqrt{x}(9+2\sqrt{x^3})^4 dx$

e $\int 6 \cdot x\sqrt[3]{x^2+4}\,dx$ f $\int \frac{2x+3}{(x^2+3x+1)^3}dx$

2. Find the antiderivatives of the following.

a $2xe^{x^2+1}$ b $\frac{3}{\sqrt{x}}e^{\sqrt{x}}$

c $\sec^2 3x e^{\tan 3x}$ d $(2ax+b)e^{-(ax^2+bx)}$

e $3\sin\frac{1}{2}xe^{\cos\frac{1}{2}x}$ f $\frac{4}{x^2}e^{4+x^{-1}}$

3. Find the antiderivatives of the following.

a $2x\sin(x^2+1)$ b $\frac{5}{\sqrt{x}}\sin(\sqrt{x})$

c $\frac{2}{x^2}\cos\left(2+\frac{1}{x}\right)$ d $\sin x\sqrt{\cos x}$

e $\frac{\sin 3x}{\cos 3x}$ f $\frac{4\sec^2 3x}{1+\tan 3x}$

4. Find the antiderivatives of the following.

a $\frac{2}{4+x^2}$ b $\frac{3}{x^2+9}$

c $\frac{\sqrt{5}}{5+x^2}$ d $\frac{1}{\sqrt{25-x^2}}$

5. Find the following indefinite integrals.

a $\int \frac{3}{1+x^2}dx$ b $\int \frac{5}{\sqrt{1-x^2}}dx$

c $\int \frac{1}{\sqrt{4-x^2}}dx$ d $\int \frac{1}{\sqrt{9-x^2}}dx$

6. Evaluate:

a $\int_1^4 x^{1/2}(1+x^{3/2})^5 dx$ b $\int_0^1 \frac{e^x}{\sqrt{1+e^x}}dx$

c $\int_0^{\frac{3\pi}{4}} \frac{3\sin x}{1+\cos x}dx$ d $\int_0^{\pi} \frac{8}{4+x^2}dx$

e $\int_{-1}^{1} e^x\cos(e^x)dx$ f $\int_0^{\frac{\pi}{2}} \sqrt{x}\sin x^{3/2}dx$

g $\int_0^{\frac{\pi}{4}} \sqrt{\tan x}\sec^2 x\,dx$ h $\int_{-1}^{1} 3x^2e^{x^3}dx$

i $\int_e^{e^2} \frac{1}{x\ln x}dx$ j $\int_3^4 x\sqrt{x^2-9}\,dx$

Extra questions

# Integrals from Derivatives

## Derivative of Inverse Trigonometric Functions

In this section, over an appropriate domain, either expression $\text{Sin}^{-1}(x)$ or $\arcsin(x)$ can be used. Similarly we can use for $\text{Cos}^{-1}(x)$ and $\arccos(x)$ as well as $\text{Tan}^{-1}(x)$ and $\arctan(x)$. That is,

$\text{Sin}^{-1}(x) = \arcsin(x), -1 \le x \le 1$,

$\text{Cos}^{-1}(x) = \arccos(x), -1 \le x \le 1$,

$\text{Tan}^{-1}(x) = \arctan(x), -\infty < x < \infty$.

It is important to keep track of how the domain of some functions is not the same as that of their derived function. For example, although the function $y = \arcsin(x)$ is defined for $-1 \le x \le 1$ its derived function, $\frac{dy}{dx}$ is defined for $-1 < x < 1$. i.e. the end points, $x = \pm 1$ are not included.

**Derivative of $\text{Sin}^{-1}(x)$**

By definition, $\text{Sin}^{-1}(x)$ is defined for $x \in [-1, 1]$. We start by letting $y = \text{Sin}^{-1}(x), -1 \le x \le 1$.

Then we have that $y = \text{Sin}^{-1}(x) \Leftrightarrow x = \sin y, -\frac{\pi}{2} \le y \le \frac{\pi}{2}$.

So, $\frac{dx}{dy} = \cos y, -\frac{\pi}{2} \le y \le \frac{\pi}{2} \Rightarrow \frac{dy}{dx} = \frac{1}{\cos y}, -\frac{\pi}{2} < y < \frac{\pi}{2}$

Note the change in domains!

Now we express $\cos y$ back in terms of $x$: Using the identity $\cos^2 y + \sin^2 y = 1$ we have:

$$\cos^2 y = 1 - \sin^2 y \therefore \cos y = \pm\sqrt{1 - \sin^2 y} = \pm\sqrt{1 - x^2}$$

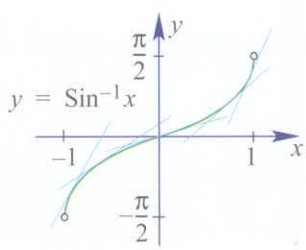
So, at this stage, the derivative of $\text{Sin}^{-1}(x)$ is given by:

$$\frac{dy}{dx} = \pm\frac{1}{\sqrt{1 - x^2}}, -1 < x < 1.$$

However, over the interval $-\frac{\pi}{2} < y < \frac{\pi}{2}$ we have that $\cos y$ is positive and so we only use the positive square root.

We then have the result that $\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}, -1 < x < 1$.

Note that we could have arrived at the same conclusion about the sign of the derivative by looking at the graph of $\text{Sin}^{-1}(x)$ for $x \in (-1, 1)$.




Using the graph of $\text{Sin}^{-1}(x)$ for $x \in (-1, 1)$, we can see that over the given interval the gradient anywhere on the curve is always positive and so we have to choose the positive square root.

Derivative of $y = \text{Sin}^{-1}\left(\frac{x}{a}\right), -a \le x \le a$ **where $a > 0$**

Using the chain rule for $y = \text{Sin}^{-1}\left(\frac{x}{a}\right), -a \le x \le a$ we set

$u = \frac{x}{a} \Rightarrow y = \text{Sin}^{-1}u, -1 \le u \le 1$, which then gives:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{\sqrt{1 - u^2}} \times \frac{1}{a} = \frac{a}{\sqrt{a^2 - x^2}} \times \frac{1}{a} = \frac{1}{\sqrt{a^2 - x^2}}$$

$-1 < \frac{x}{a} < 1$.

Note: $\sqrt{1 - u^2} = \sqrt{1 - \left(\frac{x}{a}\right)^2} = \sqrt{\frac{a^2 - x^2}{a^2}} = \frac{\sqrt{a^2 - x^2}}{a}$

$$\therefore \frac{1}{\sqrt{1 - u^2}} = \frac{a}{\sqrt{a^2 - x^2}}$$

$$\frac{d}{dx}\left(\text{Sin}^{-1}\frac{x}{a}\right) = \frac{1}{\sqrt{a^2 - x^2}}, -a < x < a$$

**Derivative of $\text{Cos}^{-1}(x)$**

Starting with the principal cosine function $f(x) = \cos x, 0 \le x \le \pi$ we define the inverse cosine function, $f^{-1}(x)$ as $f^{-1}(x) = \text{Cos}^{-1}(x), -1 \le x \le 1$. Letting $y = f^{-1}(x)$ we have $y = \text{Cos}^{-1}(x), -1 \le x \le 1$ so that $x = \cos y, 0 \le y \le \pi$. Differentiating both sides with respect to $y$ we have

$$\frac{dx}{dy} = -\sin y, 0 \le y \le \pi \Rightarrow \frac{dy}{dx} = -\frac{1}{\sin y}, 0 < y < \pi$$

Note the change in domains.

Using the trigonometric identity $\sin^2 y = 1 - \cos^2 y$ we have $\sin y = \pm\sqrt{1 - \cos^2 y} = \pm\sqrt{1 - x^2}$.

Therefore, $\frac{dy}{dx} = -\frac{1}{\pm\sqrt{1 - x^2}}, -1 < y < 1$.

We now need to determine which sign to choose. Using the graph of $y = \text{Cos}^{-1}(x), -1 < x < 1$ we see that, over this domain, the gradient is always negative and so we choose $\sqrt{1 - x^2}$.

$$\frac{d}{dx}(\text{Cos}^{-1}x) = -\frac{1}{\sqrt{1 - x^2}}, -1 < x < 1$$

and

$$\frac{d}{dx}\left(\text{Cos}^{-1}\frac{x}{a}\right) = -\frac{1}{\sqrt{a^2 - x^2}}, -a < x < a$$

**Derivative of $\text{Tan}^{-1}(x)$**

Again, we start with a principal tangent function $f(x) = \tan x, -\frac{\pi}{2} < x < \frac{\pi}{2}$.

We define the inverse tangent function, $f^{-1}(x)$ as $f^{-1}(x) = \text{Tan}^{-1}(x), -\infty < x < \infty$.

Letting $y = f^{-1}(x)$ we have $y = \text{Tan}^{-1}(x), -\infty < x < \infty$ so that $x = \tan y, -\frac{\pi}{2} < y < \frac{\pi}{2}$.

Differentiating both sides with respect to $y$ we have

$$\frac{dx}{dy} = \sec^2 y, -\frac{\pi}{2} < y < \frac{\pi}{2} \le \frac{dy}{dx} = \frac{1}{\sec^2 y}, -\frac{\pi}{2} < y < \frac{\pi}{2}.$$

However, $\tan^2 y + 1 = \sec^2 y$, therefore:

$$\frac{dy}{dx} = \frac{1}{1 + \tan^2 y}, -\frac{\pi}{2} < y < \frac{\pi}{2} = \frac{1}{1 + x^2}, -\infty < x < \infty.$$

$$\frac{d}{dx}(\text{Tan}^{-1}x) = \frac{1}{1 + x^2}, -\infty < x < \infty$$

and

$$\frac{d}{dx}\left(\text{Tan}^{-1}\frac{x}{a}\right) = \frac{a}{a^2 + x^2}, -\infty < x < \infty$$

**Example E.10.3**

Differentiate the following and specify the domain of the derivative. State the resulting anti-derivative.

a $\arcsin\left(\frac{x}{2}\right)$  b $\arctan(x+2)$

c $\arccos(x^2-9)$

a Let $u = \frac{x}{2}$ so that $f(x) = \arcsin(u)$ and as $-1 \le u \le 1 \Rightarrow -2 \le x \le 2$.

By the chain rule: $f'(x) = \frac{d}{du}\arcsin(u).\frac{du}{dx}$

$$= \frac{1}{\sqrt{1-u^2}} \times \frac{1}{2}$$

$$= \frac{1}{\sqrt{1-\left(\frac{x}{2}\right)^2}} \times \frac{1}{2}$$

$$= \frac{1}{\sqrt{4-x^2}}$$

It follows that: $\int \frac{1}{\sqrt{4-x^2}}dx = \arcsin\left(\frac{x}{2}\right) + c$

b Using the chain rule:

$$\frac{d}{dx}(\arctan[x+2]) = \frac{1}{1+[x+2]^2}, -\infty < x+2 < \infty.$$

It follows that: $\int \frac{1}{1+(x+2)^2}dx = \arctan(x+2) + c$

c Using the chain rule,

$$\frac{dy}{dx} = -\frac{2x}{\sqrt{1-[x^2-9]^2}}, x \in (-\sqrt{10}, -2\sqrt{2}) \cup (2\sqrt{2}, \sqrt{10})$$

It follows that: $\int \frac{2x}{\sqrt{1-(x^2-9)^2}}dx = \arccos(x^2-9) + c$

## Two Important Standard Integrals

The integrals that follow from these derivatives are:

$$\int \frac{1}{\sqrt{a^2-x^2}} = \mathrm{Sin}^{-1}\left(\frac{x}{a}\right)$$

$$\int \frac{a}{a^2+x^2}dx = \mathrm{Tan}^{-1}\left(\frac{x}{a}\right)$$

## Differentiating $y = a^x$

We have already considered the derivative of the natural exponential function $y = e^x$. We extend this to a more general form of the exponential function, namely, $y = a^x, a \neq 0, 1$.

The process requires an algebraic rearrangement of $y = a^x$.

Taking log (base $e$) of both sides of the equation, we have

$$y = a^x \Leftrightarrow \log_e y = \log_e a^x$$

So that, $\log_e y = x\log_e a$

Next, we differentiate both sides of the equation:

$$\frac{d}{dx}(\log_e y) = \frac{d}{dx}(x\log_e a)$$

Now (this is the tricky bit):

Using the fact that $\frac{d}{dx}(\log_e f(x)) = \frac{f'(x)}{f(x)}$ or $\frac{1}{f(x)} \times f'(x)$ and since $y$ is a function of $x$, we can write $\frac{d}{dx}(\log_e y) = \frac{1}{y} \cdot \frac{dy}{dx}$. That is, we have replaced $f(x)$ with $y$.

This means that we can now replace $\frac{d}{dx}(\log_e y) = \frac{d}{dx}(x\log_e a)$, with $\frac{1}{y} \cdot \frac{dy}{dx} = \frac{d}{dx}(x\log_e a)$.

$$\frac{d}{dx}(x\log_e a) = \log_e a \therefore \frac{1}{y} \cdot \frac{dy}{dx} = \log_e a \Leftrightarrow \frac{dy}{dx} = (\log_e a)y$$

$$= (\log_e a) \times a^x$$

i.e. if $y = a^x$ then $\frac{dy}{dx} = (\log_e a) \times a^x$

**Example E.10.4**

Differentiate the following: State the resulting anti-derivatives.

a $y = 5 \times 2^x$  b $y = 3^{4x}$  c $y = 5^{2x+1}$

a Based on our result, we have that:

$$\frac{dy}{dx} = 5 \times (\log_e 2) \times 2^x = (\log_e 32) \times 2^x .$$

It follows that: $\int \log_e 32 \times 2^x \, dx = 5 \times 2^x + c$

b Using the result that:

if $y = a^{kx}$ then $\frac{dy}{dx} = k(\log_e a) \times a^{kx}$, we have that

$$\frac{dy}{dx} = 4(\log_e 3) \times 3^{4x} = (\log_e 81) \times 3^{4x}$$

It follows that: $\int 3^{4x} \, dx = \frac{3^{4x}}{4.\log_e 3} + c$

c Letting $u = 2x + 1$ gives $y = 5^{2x+1}$ as $y = 5^u$.

Using the chain rule we have:

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = (\log_e 5) 5^u \times 2$$

$$\therefore \frac{dy}{dx} = (2\log_e 5) \times 5^{2x+1}$$

$$= (\log_e 25) \times 5^{2x+1}$$

It follows that: $\int 5^{2x-1} \, dx = \frac{5^{2x}}{10.\log_e 5} + c$

Note that from $(2\log_e 5) \times 5^{2x+1}$ a number of different acceptable answers could have been given. For example, $(2\log_e 5) \times 5^{2x+1} = (2\log_e 5) \times 5^{2x} \times 5 = 10(\log_e 5) \times 5^{2x}$.

We must not forget that we could have determined the derivative of $f(x) = a^x$ by using a first principles approach.

That is, $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{a^{x+h} - a^x}{h}$.

As $x$ is independent of the limit statement, we have:

$$f'(x) = \lim_{h \to 0} \frac{a^x(a^h - 1)}{h} = a^x \times \lim_{h \to 0} \frac{a^h - 1}{h}.$$

All that remains then is to determine $\lim_{h \to 0} \frac{a^h - 1}{h}$.

We leave this as an exercise for you. However, a starting point is to use a numerical approach, i.e. try different values of $a$ (say $a = 2$, $a = 10$) and tabulate your result for a range of (small) values of $h$ (i.e. make $h$ smaller and smaller). Then compare your numerical values to that of $\log_e 2$ for $a = 2$ and $\log_2 10$ for $a = 10$ and so on.

## Differentiating $y = \log_a x$

As in the last section, we use a simple algebraic manipulation to convert an expression for which we do not have a standard result (yet!) into one we have met before. In this case we make use of the change of base result.

i.e. given $\log_a x = \frac{\log_e x}{\log_e a}$ the equation $y = \log_a x$ can then be written as $y = \frac{1}{\log_e a} \times \log_e x$.

Now, $\frac{1}{\log_e a}$ is a real constant, and so, we are in fact differentiating an expression of the form $y = k \times \log_e x$, where $k = \frac{1}{\log_e a}$.

However if $y = k \times \log_e x$ then $\frac{dy}{dx} = k \times \frac{1}{x}$, meaning that we then have:

**If $y = \log_a x$ then $\frac{dy}{dx} = \frac{1}{\log_e a} \times \frac{1}{x}$**

### Example E.10.5

Find the derivative of:

a $\log_2 x$ b $\log_{10}(2x - 1)$ c $y = \log_4 \tan 8x$

a Given that if $y = \log_a x$ then $\frac{dy}{dx} = \frac{1}{\log_e a} \times \frac{1}{x}$ then for $y = \log_2 x$ i.e. $a = 2$ we have that

$$\frac{dy}{dx} = \frac{1}{\log_e 2} \times \frac{1}{x} = \frac{1}{(\log_e 2)x}.$$

b This time we start by letting $u = 2x - 1$ so that $y = \log_{10}(2x - 1) = \log_{10} u$.

Then, combining the chain rule with the results above (i.e. $a = 10$) we have:

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \left(\frac{1}{\log_e 10} \times \frac{1}{u}\right) \times 2$$

$$= \frac{2}{(\log_e 10)(2x - 1)}$$

c Again we combine the chain rule with the results of this section, where in this case, $a = 4$.

Let $u = \tan 8x \Rightarrow \frac{du}{dx} = 8\sec^2 8x$, then

$y = \log_4 u \Rightarrow \frac{dy}{du} = \frac{1}{\ln 4} \times \frac{1}{u}$.

Therefore, $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \left(\frac{1}{\ln 4} \times \frac{1}{u}\right) \times 8\sec^2 8x = \frac{8\sec^2 8x}{(\ln 4)\tan 8x}$

$= \frac{8}{(\ln 4)\cos 8x \sin 8x}$

$\left[\text{using } \cos 8x \sin 8x = \frac{1}{2}\sin 16x\right] \quad = \frac{8}{(\ln 2)\sin 16x}$

## Exercise E.10.2

1. Find anti-derivatives, with respect to $x$ of these functions:

a $\frac{2}{4x^2+1}$ b $\frac{1}{\sqrt{9-x^2}}$

c $\frac{-2}{\sqrt{1-4x^2}}$ d $\frac{4}{\sqrt{1-16x^2}}$

e $\frac{2}{x^2+4}$ f $\frac{1}{\sqrt{2x-x^2}}$

g $\frac{-1}{\sqrt{16-x^2}}$ h $\frac{1}{\sqrt{4-(x+1)^2}}$

i $\frac{1}{(4-x)^2+1}$ j $\frac{-1}{\sqrt{4x-x^2}}$

k $\frac{6}{4x^2+9}$ l $\frac{-1}{\sqrt{-x^2+x+2}}$

2. Evaluate:

a $\int_0^1 \frac{1}{x^2+1}dx$ b $\int_0^1 \frac{1}{\sqrt{4-x^2}}dx$

c $\int_{-1}^1 \frac{1}{4x^2+9}dx$ d $\int_{-1}^1 \frac{1}{\sqrt{27-9x^2}}dx$

e $\int_0^1 \frac{dx}{\sqrt{x-x^2}}$ Hint: $\frac{d}{dx}\left(\arcsin\sqrt{x}\right)$

f $\int_0^1 \frac{x.dx}{\sqrt{1-x^4}}$ Hint: $\frac{d}{dx}\left(\arccos x^2\right)$

g $\int_0^1 \frac{x.dx}{1+x^4}$ Hint: $\frac{d}{dx}\left(\arctan x^2\right)$

h $\int_0^1 \left(\frac{1}{1+x^2} + 2x - 3e^{-2x}\right)dx$

3. Differentiate the following and state the resulting anti-derivative

a $y = 4^x$ b $y = 3^x$

c $y = 8^x$ d $y = 3 \times 5^x$

e $y = 7 \times 6^x$ f $y = 2 \times 10^x$

g $y = 6^{x-2}$ h $y = 2^{3x+1}$

i $y = 5 \times 7^{3-x}$

4. Find:

a $\int 2^x dx$ b $\int 3^{2x} dx$

c $\int 2^{2x-4} dx$ d $\int 3^{4x-5} dx$

e $\int 3^{x/2} dx$ f $\int 3^{3x/4} dx$

5. Evaluate:

a $\int_0^3 2^x dx$ b $\int_{-1}^3 3^x dx$

c $\int_{-1}^3 3^{2x} dx$ d $\int_{-2}^3 2^{3-x} dx$

e $\int_1^3 \left(2^{3x} + 2x - 3\right)dx$

6. Find the value of $A$ such that $\int_0^A 2^{Ax} dx = \frac{511}{3.\log_e 2}$.

7. Find the value of $B$ such that $\int_0^1 \frac{B}{1+x^2} dx = \frac{5\pi}{4}$

8. Find the value of $C$ such that $\int_0^C \frac{1}{\sqrt{4-x^2}} dx = \frac{\pi}{6}$

# Partial Fractions

In Chapter A.6 we met the technique of splitting algebraic fractions into simpler expressions known as **partial fractions**. This can be a very useful tool in finding anti-derivatives.

This is particularly the case when combined with the standard form the results from this derivative:

$$y=\log_e(f(x))\Rightarrow\frac{dy}{dx}=\frac{f'(x)}{f(x)}$$

This result follows from the Chain Rule and leads to the standard form:

$$\int\left(\frac{f'(x)}{f(x)}\right)dx=\log_e(f(x))+c$$

**Example E.10.6**

Find:

a $\int\left(\frac{2x-4}{x^2-4x}\right)dx$ b $\int(\tan(x))dx$

a $\int\left(\frac{2x-4}{x^2-4x}\right)dx$

Observe that: $\frac{d}{dx}(x^2-4x)=2x-4$ so that:

$$\int\left(\frac{2x-4}{x^2-4x}\right)dx=\log_e(x^2-4x)+c$$

We can check this by Chain Rule differentiation:

$$y=\log_e(x^2-4x)$$

$$u=x^2-4x\Rightarrow\frac{du}{dx}=2x-4$$

$$y=\log_e(u)\Rightarrow\frac{dy}{du}=\frac{1}{u}$$

$$\frac{dy}{dx}=\frac{du}{dx}\times\frac{dy}{du}\Rightarrow\frac{dy}{dx}=(2x-4)\times\frac{1}{u}=\frac{2x-4}{x^2-4x}$$

b
$$\int\tan(x)dx=\int\frac{\sin(x)}{\cos(x)}dx$$
$$=-\int\frac{-\sin(x)}{\cos(x)}dx$$
$$=-\log_e(\cos(x))+c$$

We can now set about using these two techniques to tackle some quite complex integrals.

**Example E.10.7**

Find:

a $\int\frac{1}{x^2-x-6}dx$

b $\int\frac{8x^2-22x+8}{(x-1)(2x^2-6x+1)}dx$

a The integrand is not in the form 'top the derivative of the bottom'. However, we can split it into partial fractions:

$$\frac{1}{x^2-x-6}\equiv\frac{1}{(x+2)(x-3)}$$
$$\equiv\frac{A}{x+2}+\frac{B}{x-3}$$

Using techniques developed in Chapter A.6, we now find $A$ & $B$.

$$\frac{1}{x^2-x-6}\equiv\frac{A(x-3)+B(x+2)}{(x+2)(x-3)}$$

Since the denominators are identical, so are the numerators:

$$A(x-3)+B(x+2)\equiv1$$

Equating coefficients:

$$x^1{:}A+B=0...[1]$$
$$x^0{:}-3A+2B=1...[2]$$

$$2.[1]-[2]{:}5A=-1\Rightarrow A=-\frac{1}{5},B=\frac{1}{5}$$

The integral can now be split into two parts, both of which are 'log form':

$$\int\frac{1}{x^2-x-6}dx=-\frac{1}{5}\int\frac{dx}{x+2}+\frac{1}{5}\int\frac{dx}{x-3}$$
$$=-\frac{1}{5}\log_e(x+2)+\frac{1}{5}\log_e(x-3)+c$$
$$=\frac{1}{5}\log_e\left(\frac{x-3}{x+2}\right)+c$$

b Remember that partial fractions only 'work' if their rules are followed. The first is that the numerator must have lower degree than the denominator.

If not, polynomial division must occur first and the remainder term split. The other rule is that constants in the numerator only work for linear factors in the denominator. In this case, we do not need to divide first, but we must be careful with the choice of numerators.

$$\frac{8x^2-22x+8}{(x-1)(2x^2-6x+1)} \equiv \frac{A}{x-1}+\frac{Bx+C}{2x^2-6x+1}$$

The solution can now proceed:

$$\frac{8x^2-22x+8}{(x-1)(2x^2-6x+1)} \equiv \frac{A(2x^2-6x+1)+(Bx+C)(x-1)}{(x-1)(2x^2-6x+1)}$$

It follows that:

$$\begin{aligned} 8x^2-22x+8 &\equiv A(2x^2-6x+1)+(Bx+C)(x-1) \\ &\equiv (2A+B)x^2+(-6A-B+C)x+(A-C) \end{aligned}$$

Equating coefficients:

$$\begin{aligned} &x^2: 2A+B=8 \ldots[1] \\ &x: -6A-B+C=-22 \ldots[2] \\ &x^0: A-C=8 \ldots[3] \\ &[2]+[3]: -5A-B=-14 \ldots[4] \\ &[1]+[4]: -3A=-6 \Rightarrow A=2 \\ &[1]: 2\times 2+B=8 \Rightarrow B=4 \\ &[3]: 2-C=8 \Rightarrow C=-6 \end{aligned}$$

The integral can now be split:

$$\int \frac{8x^2-22x+8}{(x-1)(2x^2-6x+1)}dx = \int \frac{2dx}{x-1}+\int \frac{4x-6}{2x^2-6x+1}dx$$

Both of these are 'log form' and we can proceed straight to the answer:

$$\int \frac{2dx}{x-1}+\int \frac{4x-6}{2x^2-6x+1}dx = 2\log_e(x-1)+\log_e(2x^2-6x+1)+c$$

The technique can also be applied to definite integrals

**Example E.10.8**

Find exact values for:

a $\displaystyle\int_1^2 \frac{4dx}{(x+3)(x+5)}$

b $\displaystyle\int_{-1}^1 \frac{(2x^2+3x-34)dx}{(x+5)}$

$$\frac{4}{(x+3)(x+5)} \equiv \frac{A}{x+3}+\frac{B}{x+5}$$

$$\begin{aligned} &4 \equiv A(x+5)+B(x+3) \\ &x^0: 5A+3B=4 \ldots[1] \\ &x^1: A+B=0 \ldots[2] \\ &[1]-3.[2]: 2A=4 \Rightarrow A=2 \\ &[2]: 3+B=0 \Rightarrow B=-2 \end{aligned}$$

The definite integral can now be calculated:

$$\begin{aligned} \int_1^2 \frac{4dx}{(x+3)(x+5)} &= \int_1^2 \frac{2dx}{(x+3)} - \int_1^2 \frac{2dx}{(x+5)} \\ &= 2[\log_e(x+3)]_1^2 - 2[\log_e(x+5)]_1^2 \\ &= 2(\log_e(5)-\log_e(4)) - 2(\log_e(7)-\log_e(6)) \\ &= 2\log_e\left(\frac{5\times 6}{4\times 7}\right) \\ &= 2\log_e\left(\frac{15}{14}\right) \end{aligned}$$

b $\displaystyle\int_{-1}^1 \frac{(2x^2+3x-34)dx}{(x+5)}$

On this occasion, as the numerator is of higher degree than the denominator, we must divide first.

$$\begin{array}{r|l} & 2x \quad -7 \\ \hline x+5 & 2x^2+3x-34 \\ & 2x^2+10x \\ & \quad -7x-34 \\ & \quad -7x-35 \\ & \qquad\quad 1 \end{array}$$

$$\begin{aligned} \int_{-1}^1 \frac{(2x^2+3x-34)dx}{(x+5)} &= \int_{-1}^1 \left(2x-7+\frac{1}{(x+5)}\right)dx \\ &= \left[x^2-7x+\log_e(x+5)\right]_{-1}^1 \\ &= 1-7+\log_e 6-(1+7+\log_e 4) \\ &= -14+\log_e\left(\frac{3}{2}\right) \end{aligned}$$

## Exercise E.10.3

1. Find:

a $\int \frac{dx}{(x-1)(x+3)}$

b $\int \frac{dx}{(x-2)(x+4)}$

c $\int \frac{dx}{(2x-1)(x+3)}$

d $\int \frac{dx}{(2x-5)(3x+1)}$

e $\int \frac{dx}{(3x-1)(2x+1)(x-4)}$

f $\int \frac{x^3 dx}{(x-1)}$

g $\int \frac{x^3 dx}{(x^2-1)}$

h $\int \frac{x^3 dx}{(x^2+1)}$

i $\int \frac{4x^3-13x^2-10x+34}{(x-4)(x+2)(2x-1)}dx$

j $\int \frac{x^3}{(x+1)(x-3)}dx$

2. Evaluate:

a $\int_0^1 \frac{dx}{x+2}$

b $\int_1^3 \frac{dx}{(x+2)(2x-7)}$

c $\int_0^1 \frac{dx}{(2x+1)(x^2-4)}$

d $\int_0^2 \frac{x^2+4}{(x+2)(2x+1)}dx$

e $\int_0^2 \frac{x^2+3x+1}{(x+2)(2x+1)(x-3)}dx$

f $\int_{-1}^3 \frac{5}{4+x^2}dx$

3. Find the value of $A$ such that $\int_A^3 \frac{3}{4+x}dx = 3.\log_e\left(\frac{7}{6}\right)$.

4. Use the equation of a unit circle: $x^2+y^2=1$ to prove that the area of the circle is $\pi$.

5. Find the area between the curve with equation:

$y = \frac{2}{(x+2)(x+3)}$ and the lines $x = 1$ and $x = 2$.

6. Find the closed area bounded by the line $y = 1 - \frac{x}{2}$ and the curve $y = \frac{1}{x+1}$.

7. Find the value of B such that:

$$\int_A^{2A} \frac{dx}{(2x-1)(x+3)} = \frac{1}{7}\log_e\left(\frac{22}{15}\right)$$

8. Find, correct to 4 significant figures, the value of:

$$\int_0^1 \frac{3.sin(x)}{cos(x)+2}dx .$$

## Substitution Rule

In the previous section, we considered integrals that required the integrand to be of a particular form in order to carry out the antidifferentiation process.

For example, the integral $\int 2x\sqrt{1+x^2}dx$ is of the form $\int h'(x)[h(x)]^n dx$ and so we could proceed by using the result:

$$\int h'(x)[h(x)]^n dx = \frac{1}{n+1}[h(x)]^{n+1} + c.$$

Next consider the integral $\int x\sqrt{x-1}dx$. This is not in the form $\int h'(x)[h(x)]^n dx$ and so we cannot rely on the recognition approach we have used so far. To determine such an integral we need to use a formal approach.

Indefinite integrals that require the use of the general power rule can also be determined by making use of a method known as the substitution rule (or change of variable rule). The name of the rule is indicative of the process itself. We introduce a new variable, $u$ (say), and substitute it for an appropriate part (or the whole) of the integrand. An important feature of this method is that it will enable us to find the integral of expressions that cannot be determined by the use of the general power rule.

We illustrate this process using a number of examples (remembering that the success of this method is in making the appropriate substitution). The basic steps in integration by substitution can be summarized as follows:

1. Define $u$ (i.e. let $u$ be a function of the variable which is part of the integrand).
2. Convert the integrand from an expression in the original variable to an expression in $u$ (this means that you also need to convert the '$dx$' term to a '$du$' term – if the original variable is $x$).
3. Integrate and then rewrite the answer in terms of $x$ (by substituting back for $u$).

NB: This is only a guide, you may very well skip steps or use a slightly different approach.

**Example E.10.9**

a $\int (2x+1)^4 dx$

b $\int 2x(x^2+1)^3 dx$ c $\int \frac{x^2}{\sqrt{x^3-4}}dx$.

a Although this integral can be evaluated by making use of the general power rule, we use the substitution method to illustrate the process:

In this case we let $u = 2x+1 \Rightarrow \frac{du}{dx} = 2 \therefore dx = \frac{1}{2}du$.

Having chosen $u$, we have also obtained an expression for $dx$ and we are now in a position to carry out the substitution for the integrand:

$$\int (2x+1)^4 dx = \int u^4 \times \left(\frac{1}{2}du\right) = \frac{1}{2}\int u^4 du = \frac{1}{2} \times \frac{1}{5}u^5 + c$$

$$= \frac{1}{10}u^5 + c$$

Substituting back, we obtain, in terms of $x$: $= \frac{1}{10}(2x+1)^5 + c$

b This time, we let $u = x^2+1$. Note the difference between this substitution and the one used in part a. We are making a substitution for a non-linear term.

Now, $u = x^2+1 \Rightarrow \frac{du}{dx} = 2x \therefore \frac{1}{2}du = xdx$.

Although there is an $x$ attached to the $dx$ term, hopefully, when we carry out the substitution, everything will fall into place.

Now, $\int 2x(x^2+1)^3 dx = \int 2(x^2+1)^3 x dx$

(We have moved the $x$ next to the $dx$.)

$= \int 2u^3 \times \frac{1}{2}du$ (substituting $xdx$ for ½$du$.)

$= \frac{1}{4}u^4 + c$

$= \frac{1}{4}(x^2+1)^4 + c$

NB: A second (alternate) method is to obtain an expression for $dx$ in terms of one or both variables. Make the substitution and then simplify. Although there is some dispute as to the 'validity' of this method, in essence it is the same. We illustrate this next.

c Let $u = x^3 - 4 \Rightarrow \frac{du}{dx} = 3x^2 \therefore dx = \frac{1}{3x^2}du$, making the substitution for $u$ and $dx$, we have:

$\int \frac{x^2}{\sqrt{x^3-4}}dx = \int \frac{x^2}{\sqrt{u}} \times \frac{1}{3x^2}du = \frac{1}{3}\int u^{-1/2}du$ (Notice the $x^2$ terms cancel!)

$$= \frac{2}{3}u^{1/2} + c$$

$$= \frac{2}{3}\sqrt{x^3-4} + c$$

### Example E.10.10

Find $\int x\sqrt{x-1}\,dx$.

Letting $u = x - 1 \Rightarrow \frac{du}{dx} = 1 \therefore du = dx$.

This then gives $\int x\sqrt{x-1}\,dx = \int x\sqrt{u}\,du$.

We seem to have come at an impasse. After carrying out the substitution we are left with two variables, $x$ and $u$, and we need to integrate with respect to $u$. This is a type of integrand where not only do we substitute for the $x - 1$ term, but we must also substitute for the $x$ term that has remained as part of the integrand, from $u = x - 1$ we have $x = u + 1$.

Therefore:

$$\int x\sqrt{x-1}\,dx = \int x\sqrt{u}\,du = \int (u+1)u^{1/2}du$$

$$= \int (u^{3/2} + u^{1/2})du$$

$$= \frac{2}{5}u^{5/2} + \frac{2}{3}u^{3/2} + c$$

$$= \frac{2}{5}\sqrt{(x-1)^5} + \frac{2}{3}\sqrt{(x-1)^3} + c$$

### Example E.10.11

The gradient at any point on the curve $y = f(x)$ is given by the equation: $\frac{dy}{dx} = \frac{1}{\sqrt{x+2}}$. The curve passes through the point (2, 3). Find the equation of this curve.

Integrating both sides of $\frac{dy}{dx} = \frac{1}{\sqrt{x+2}}$ with respect to $x$, we have:

$$\int \frac{dy}{dx}dx = \int \frac{1}{\sqrt{x+2}}dx.$$

Let $u = x + 2 \Rightarrow \frac{du}{dx} = 1 \therefore du = dx$.

So, $\int \frac{1}{\sqrt{x+2}}dx = \int \frac{1}{\sqrt{u}}du = \int u^{-1/2}du = 2\sqrt{u} + c$

Therefore, we have $y = f(x) = 2\sqrt{x+2} + c$.

Now, $f(2) = 3 \Rightarrow 3 = 2\sqrt{4} + c \Leftrightarrow c = -1$.

Therefore, $f(x) = 2\sqrt{x+2} - 1$.

### Example E.10.12

Find the indefinite integral of the following.

a $x^2e^{x^3+4}$ b $e^x\cos(e^x)$

c $\frac{3x}{x^2+4}$ d $x^2\sqrt{x+1}$.

a Let $u = x^3 + 4 \Rightarrow \frac{du}{dx} = 3x^2 \therefore \frac{1}{3x^2}du = dx$.

Substituting, we have:

$$\int x^2e^{x^3+4}dx = \int x^2e^u \times \frac{1}{3x^2}du = \frac{1}{3}\int e^u du$$

$$= \frac{1}{3}e^u + c$$

$$= \frac{1}{3}e^{x^3+4} + c$$

b Let $u = e^x \Rightarrow \frac{du}{dx} = e^x \therefore dx = \frac{1}{e^x}du$.

Substituting, we have:

$$\int e^x\cos(e^x)dx = \int e^x\cos u \times \frac{1}{e^x}du = \int \cos u\,du$$

$$= \sin u + c$$

$$= \sin(e^x) + c$$

c Let $u = x^2+4 \Rightarrow \frac{du}{dx} = 2x \therefore dx = \frac{1}{2x}du$.

Substituting, we have:

$$\int \frac{3x}{x^2+4}dx = \int \frac{3x}{u} \times \frac{1}{2x}du = \frac{3}{2}\int \frac{1}{u}du = \frac{3}{2}\ln u + c$$

$$= \frac{3}{2}\ln(x^2+4) + c$$

d Let $u = x+1 \Rightarrow \frac{du}{dx} = 1 \therefore dx = du$.

Substituting, we have $\int x^2\sqrt{x+1}dx = \int x^2\sqrt{u}du$. Then, as there is still an $x$ term in the integrand, we will need to make an extra substitution. From $u = x+1$ we have $x = u-1$.

Therefore,

$$\int x^2\sqrt{u}du = \int (u-1)^2\sqrt{u}du = \int (u^2-2u+1)u^{1/2}du$$

$$= \int (u^{5/2} - 2u^{3/2} + u^{1/2})du$$

$$= \frac{2}{7}u^{7/2} - \frac{4}{5}u^{5/2} + \frac{2}{3}u^{3/2} + c$$

$$= \frac{2}{7}(x+1)^{7/2} - \frac{4}{5}(x+1)^{5/2} + \frac{2}{3}(x+1)^{3/2} + c$$

**Example E.10.13**

Find the indefinite integral of the following:

a $\sin 3x\cos^2 3x$ b $\frac{\sin 2x}{5+\cos 2x}$ c $\frac{\arctan x}{x^2+1}$.

a Let $u = \cos 3x \Rightarrow \frac{du}{dx} = -3\sin 3x \therefore dx = -\frac{1}{3\sin 3x}du$.

Substituting, we have:

$$\int \sin 3x\cos^2 3x dx = \int \sin 3x u^2 \times -\frac{1}{\sin 3x}du$$

$$= -\frac{1}{3}\int u^2 du$$

$$= -\frac{1}{3}\cdot\frac{1}{3}u^3 + c$$

$$= -\frac{1}{9}\cos^3 3x + c$$

b Let $u = 5+\cos 2x \Rightarrow \frac{du}{dx} = -2\sin 2x \therefore dx = -\frac{1}{2\sin 2x}du$.

Substituting, we have $\int \frac{\sin 2x}{5+\cos 2x}dx = \int \frac{\sin 2x}{u} \times -\frac{1}{2\sin 2x}du$

$$= -\frac{1}{2}\int \frac{1}{u}du$$

$$= -\frac{1}{2}\ln u + c$$

$$= -\frac{1}{2}\ln(5+\cos 2x) + c$$

c Let $u = \arctan x \Rightarrow \frac{du}{dx} = \frac{1}{1+x^2} \therefore dx = (1+x^2)du$.

Substituting, we have:

$$\int \frac{\arctan x}{x^2+1}dx = \int \frac{u}{x^2+1} \times (1+x^2)du = \int u du$$

$$= \frac{1}{2}u^2 + c$$

$$= \frac{1}{2}(\arctan x)^2 + c$$

**Example E.10.14**

Evaluate:

a $\int_1^2 xe^{x^2}dx$ b $\int_1^3 \sqrt{2x+3}dx$ c $\int_0^2 \sqrt{4-x^2}.dx$

When using the substitution method to evaluate a definite integral, it is generally more efficient to transform the terminals (limits) of the integral as well as the integrand. This process is illustrated by the following examples.

a This is solved using the substitution $u = x^2, \frac{du}{dx} = 2x$

The integrand is transformed to:

$$\int xe^{x^2}dx = \int \frac{1}{2}e^u du = \frac{1}{2}e^{x^2} + c$$

Having established that the substitution will work, we can now use it to transform the terminals.

The lower terminal is $x = 1 \Rightarrow u = 1^2 = 1$ and the upper terminal is $x = 2 \Rightarrow u = 2^2 = 4$.

Thus: $\int_1^2 xe^{x^2}dx = \int_1^4 \frac{1}{2}e^u du = \frac{1}{2}[e^u]_1^4 = \frac{1}{2}(e^4 - e)$

b Use $u = 2x+3, \frac{du}{dx} = 2$ and:

$x = 1 \Rightarrow u = 5, x = 3 \Rightarrow u = 9$

$$\int_1^3 \sqrt{2x+3}dx = \int_5^9 \frac{1}{2}u^{1/2}du = \frac{1}{2}\left[\frac{2}{3}u^{3/2}\right]_5^9$$
$$= \frac{1}{3}(9^{3/2} - 5^{3/2})$$
$$= \frac{1}{3}(27 - 5\sqrt{5})$$

c Let $x = 2\sin\theta, \frac{dx}{d\theta} = 2\cos\theta$.

The terminals transform to:

$x = 0 \Rightarrow 0 = 2\sin\theta \Rightarrow \theta = 0$

$x = 2 \Rightarrow 2 = 2\sin\theta \Rightarrow \theta = \frac{\pi}{2}$.

$$\therefore \int_0^2 \sqrt{4-x^2}dx = \int_0^{\frac{\pi}{2}} \sqrt{4 - 4\sin^2\theta} \times 2\cos\theta d\theta$$
$$= \int_0^{\frac{\pi}{2}} 2\sqrt{1-\sin^2\theta} \times 2\cos\theta d\theta$$
$$= 2\int_0^{\frac{\pi}{2}} 2\cos^2\theta d\theta$$
$$= 2\int_0^{\frac{\pi}{2}} (1 + \cos 2\theta) d\theta$$
$$= 2\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{2}}$$
$$= \pi$$

## Exercise E.10.4

1. Find the following, using the given $u$ substitution.

a $\int 2x\sqrt{x^2+1}dx$, $u = x^2+1$

b $\int 3x^2\sqrt{x^3+1}dx$, $u = x^3+1$

c $\int 2x^3\sqrt{4-x^4}dx$, $u = 4-x^4$

d $\int \frac{3x^2}{x^3+1}dx$, $u = x^3+1$

e $\int \frac{x}{(3x^2+9)^4}dx$, $u = 3x^2+9$

f $\int 2xe^{x^2+4}dx$, $u = x^2+4$

g $\int \frac{2z+4}{z^2+4z-5}dz$, $u = z^2+4z-5$

2. Using the substitution method, find:

a $\int x\sqrt{2x-1}dx$ b $\int x^2\sqrt{1-x}dx$

c $\int (x+1)\sqrt{x-1}dx$ d $\int \sec^2 x e^{\tan x}dx$

3. Using an appropriate substitution, evaluate the following, giving exact values.

a $\int_{-1}^{1} \frac{2x}{x^2+1}dx$ b $\int_0^1 \frac{2x^2}{x^3+1}dx$

c $\int_{10}^{12} \frac{2x+1}{x^2+x-2}dx$ d $\int_0^{\frac{\pi}{2}} \frac{\cos x}{1+\sin x}dx$

4. Using an appropriate substitution, evaluate the following, giving exact values.

a $\int_1^2 x\sqrt{x^2+3}dx$ b $\int_0^{\frac{\pi}{2}} 3x\sin(4x^2+\pi)dx$

c $\int_{-1}^{1} (3x+2)^4 dx$ d $\int_{-2}^{1} \frac{1}{x+3}dx$

5. Using an appropriate substitution, find the following, giving exact values where required.

a $\int_0^{\frac{\pi}{2}} \sin^3 x\cos x dx$ b $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin x\sec^2 x dx$

c $\int_0^{\frac{\pi}{3}} \cos^3 x\sin 2x dx$ d $\int_0^{\frac{\pi}{3}} \frac{\sin 2x}{\sqrt{\cos^3 x}}dx$

6. Using an appropriate substitution, find the following, giving exact values where required.

a $\int_{-2}^{-1} x\sqrt{x+2}dx$

b $\int_{-1}^{2} x\sqrt{2-x}\,dx$

7. Find the following indefinite integrals.

a $\int \frac{1}{x^2+6x+10}dx$ b $\int \frac{1}{x^2-x+1}dx$

c $\int \frac{1}{\sqrt{1+4x-x^2}}dx$ d $\int \frac{3}{\sqrt{8-2x-x^2}}dx$

8. Given that $\frac{2-x^2}{(x^2+1)(x^2+4)} \equiv \frac{A}{x^2+1}+\frac{B}{x^2+4}$, find $A$ and $B$.

Hence show that $\int_0^1 \frac{2-x^2}{(x^2+1)(x^2+4)}dx = arctan\left(\frac{1}{3}\right)$.

9. Find $\int_0^k \frac{1}{x^2+1}dx, k>0$.

Evaluate this definite integral for:

i $k = \frac{1}{\sqrt{3}}$ ii $k = 1$.

Find $\lim_{k\to\infty}\int_0^k \frac{1}{x^2+1}dx$. Hence, find $\int_{-\infty}^{\infty} \frac{1}{x^2+1}dx$.

10. Find $\int \frac{1}{\sqrt{x}+1}dx$. Hence evaluate $\int_0^1 \frac{1}{\sqrt{x}+1}dx$.

11. If $z = cis\theta$, use the expansion of $\left(z-\frac{1}{z}\right)^4$ to show that $8\sin^4\theta = \cos 4\theta - 4\cos 2\theta + 3$.

Hence, using the substitution $x = k\sin^2\theta$, evaluate:

$\int_0^k x\sqrt{\frac{x}{k-x}}dx$, $0<\theta<\pi$.

Extra questions

# Integration by Parts

## The basics

Consider the indefinite integral $\int x\cos x\,dx$.

Applying any of the techniques we have been using so far will not help us determine the integral. Let us start the process by first finding the derivative of $x\sin x$:

$$\frac{d}{dx}(x\sin x) = \frac{d}{dx}(x)\sin x + x\frac{d}{dx}(\sin x) \text{ (using product rule)}$$

$$\therefore \frac{d}{dx}(x\sin x) = \sin x + x\cos x$$

We observe that the term $x\cos x$ has now appeared on the R.H.S. so we can write

$$x\cos x = \frac{d}{dx}(x\sin x) - \sin x$$

$$\therefore \int x\cos x\,dx = \int\left[\frac{d}{dx}(x\sin x) - \sin x\right]dx$$
$$= x\sin x + \cos x + c$$

Such a process requires considerable foresight. However, this integrand falls into a category of integrands that can be antidifferentiated via a technique known as integration by parts. The method is identical to that which we have just used in determining $\int x\cos x\,dx$.

We develop a general expression for integrands that involve a product of two functions.

Step 1: Consider the product $u(x)v(x)$.

Step 2: Using the product rule for differentiation we have:

$$\frac{d}{dx}(u(x)v(x)) = u(x)\frac{dv}{dx} + v(x)\frac{du}{dx}$$

Step 3: Integrating both sides with respect to $x$ gives:

$$u(x)v(x) = \int u(x)\frac{dv}{dx}dx + \int v(x)\frac{du}{dx}dx$$

Step 4: Rearranging, to obtain $\int u(x)\frac{dv}{dx}dx$, we have:

$$\int u(x)\frac{dv}{dx}dx = u(x)v(x) - \int v(x)\frac{du}{dx}dx$$

In the previous case, we would set, $u(x) = x$ and $\frac{dv}{dx} = \cos x$ and the result would then follow through.

The success of this technique is dependent on your ability to identify the 'correct' $u(x)$ and $v(x)$.

For example, had we used $u(x) = \cos x$ and $\frac{dv}{dx} = x$,

we would have the expression

$$\int x\cos x\,dx = \frac{1}{2}x^2\cos x - \int \frac{1}{2}x^2(-\sin x)dx$$

– which is not helpful.

We now consider some examples to highlight the process involved.

**Example E.10.15**

Evaluate:

a $\int x\cos x\,dx$  b $\int \frac{x}{3}e^{2x}dx$.

a $\int x\cos x\,dx$.

Applying the parts formula with $u(x) = x$ and $\frac{dv}{dx} = \cos x$, it follows that $v(x) = \sin x$ gives:

| $u(x)$ → Find | $\frac{du}{dx}$ |
|---|---|
| $x$ | $1$ |
| $v(x)$ ← Find | $\frac{dv}{dx}$ |
| $\sin x$ | $\cos x$ |

$$\int u(x)\frac{dv}{dx}dx = u(x)v(x) - \int v(x)\frac{du}{dx}dx$$

$$\int x\cos x\,dx = x \times \sin x - \int \sin x \times 1\,dx$$

$$= x\sin x - (-\cos x) + c$$

$$= x\sin x + \cos x + c$$

You should check that this is correct by differentiating the answer.

Many people remember the 'parts formula' by thinking of the question as consisting of two parts each of which are functions of the independent variable. One of these functions is to be integrated and the other differentiated. It pays to select a function that becomes simpler in derivative form to be the 'part' that is differentiated. Often, though not always, this will be the polynomial part.

b $\int \frac{x}{3}e^{2x}dx$.

In this case we choose the function to be differentiated as $u(x) = \frac{x}{3}$ and the function to be integrated as $\frac{dv}{dx} = e^{2x} \Rightarrow v(x) = \frac{1}{2}e^{2x}$

| $u(x)$ → Find | $\frac{du}{dx}$ |
|---|---|
| $\frac{x}{3}$ | $\frac{1}{3}$ |
| $v(x)$ ← Find | $\frac{dv}{dx}$ |
| $\frac{1}{2}e^{2x}$ | $e^{2x}$ |

$$\int u(x)\frac{dv}{dx}dx = u(x)v(x) - \int v(x)\frac{du}{dx}dx$$

$$\int \frac{x}{3}e^{2x}dx = \frac{x}{3} \times \frac{1}{2}e^{2x} - \int \frac{1}{2}e^{2x} \times \frac{1}{3}dx$$

$$= \frac{x}{6}e^{2x} - \frac{1}{12}e^{2x} + c$$

## Exercise E.10.5

1. Integrate the following expressions with respect to $x$.

   a $x\sin x$  b $x\cos\frac{x}{2}$

   c $2x\sin\frac{x}{2}$  d $xe^{-x}$

2. Use integration by parts to antidifferentiate:

   a $x\sqrt{x+1}$  b $x\sqrt{x-2}$

3. Find:

   a $\int \mathrm{Cos}^{-1}x\,dx$  b $\int \mathrm{Tan}^{-1}x\,dx$

4. Find:

a $\int x\text{Cos}^{-1}x\,dx$ b $\int x\text{Tan}^{-1}x\,dx$

5. Find:

a $\int_0^{\frac{\pi}{4}} x\sin 2x\,dx$ b $\int_0^1 xe^{2x}\,dx$

c $\int_1^{(e-1)} x\ln(x+1)\,dx$ d $\int_1^2 (x-1)\ln x\,dx$

e $\int_{\frac{\pi}{8}}^{\frac{\pi}{4}} x\cos 2x\,dx$ f $\int_1^e \frac{\ln x}{x}\,dx$

6. Find $\int_0^1 x^2\text{Tan}^{-1}x\,dx$.

7. Show that $\frac{d}{dx}[\ln(\sec x + \tan x)] = \sec x$.

Hence find $\int_0^{\frac{\pi}{4}} \sec^3 x\,dx$.

8. Find:

a $\int \cos(\ln x)\,dx$

b $\int \sin(\ln x)\,dx$

c $\int x^3\sqrt{1-x^2}$

Extra questions

## Repeated Integration by Parts

In Exercise E.10.5, Question 7 required the repeated use of integration by parts. There will be occasions on which you will need to use the 'parts' formula more than once to evaluate an integral as in the following examples.

**Example E.10.16**

Evaluate:

a $\int x^2\cos 2x\,dx$ b $\int e^{2x}\sin\frac{x}{3}\,dx$.

a
$$\int x^2\cos 2x\,dx = x^2\times\frac{1}{2}\sin 2x - \int 2x\times\frac{1}{2}\sin 2x\,dx$$
$$\left[\text{using } u = x^2,\ \frac{dv}{dx} = \cos 2x\right]$$
$$= \frac{1}{2}x^2\sin 2x - \int x\sin 2x\,dx$$
Repeated use of 'parts' formula.
$$= \frac{1}{2}x^2\sin 2x - \left(x\times -\frac{1}{2}\cos 2x - \int\left(-\frac{1}{2}\cos 2x\right)dx\right)$$
$$= \frac{1}{2}x^2\sin 2x + \frac{1}{2}x\cos 2x - \frac{1}{4}\sin 2x + c$$

b
$$\int e^{2x}\sin\frac{x}{3}\,dx = e^{2x}\times -3\cos\frac{x}{3} - \int 2e^{2x}\times -3\cos\frac{x}{3}\,dx$$
$$= -3e^{2x}\cos\frac{x}{3} - \left(-18e^{2x}\sin\frac{x}{3} - \int -36e^{2x}\sin\frac{x}{3}\,dx\right)$$
$$= -3e^{2x}\cos\frac{x}{3} + 18e^{2x}\sin\frac{x}{3} - 36\int e^{2x}\sin\frac{x}{3}\,dx$$

The required integral appears on both sides of this equation, which rearranges to:

$$37\int e^{2x}\sin\frac{x}{3}\,dx = -3e^{2x}\cos\frac{x}{3} + 18e^{2x}\sin\frac{x}{3}$$
$$\therefore \int e^{2x}\sin\frac{x}{3}\,dx = -\frac{3}{37}e^{2x}\cos\frac{x}{3} + \frac{18}{37}e^{2x}\sin\frac{x}{3} + c$$

## Exercise E.10.6

1. Find the following integrals (not all are best evaluated using the parts formula).

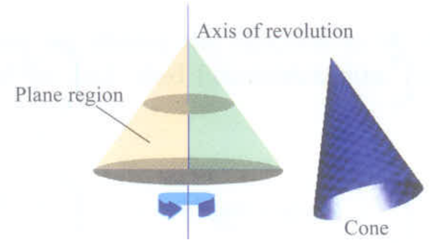
| | | | |
|---|---|---|---|
| a | $\int x^2 e^x dx$ | b | $\int 3x^2\cos(2x)dx$ |
| c | $\int x^3\ln(2x)dx$ | d | $\int e^x\sin(2x)dx$ |
| e | $\int x^2\cos(3x)dx$ | f | $\int e^{-2x}\cos(2x)dx$ |
| g | $\int 4x^3\sin\frac{x}{2}dx$ | h | $\int\frac{1}{x}\ln x dx$ |
| i | $\int(\ln(3x))^2dx$ | j | $\int\cos x\sin(2x)dx$ |
| k | $\int e^{ax}\cos\frac{x}{a}dx$ | l | $\int x^2\sqrt{x+2}dx$ |
| m | $\int x^3\ln(ax)dx$ | n | $\int\frac{x^2}{\sqrt{4-x^2}}dx$ |
| o | $\int\frac{3x^2dx}{\sqrt{x^2-9}}$ | p | $\int\frac{x}{x^2+4}dx$ |
| q | $\int\frac{x^2}{x^2+4}dx$ | | |

2. Evaluate the following.

| | | | |
|---|---|---|---|
| a | $\int_0^{\frac{\pi}{2}} x\cos^2 x dx$ | b | $\int_0^{\frac{\pi}{2}} x\sin x\cos x dx$ |
| c | $\int_{\frac{\pi}{2}}^{2\pi} e^x\cos x dx$ | d | $\int_0^{\ln 2} x^2 e^{-x}dx$ |
| e | $\int_{\frac{\pi}{b}}^{\frac{2\pi}{b}} e^{ax}\cos bx dx$ | f | $\int_1^e(\ln x)^2dx$ |

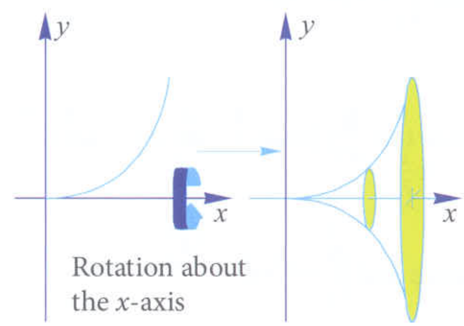
# Volumes of Revolution

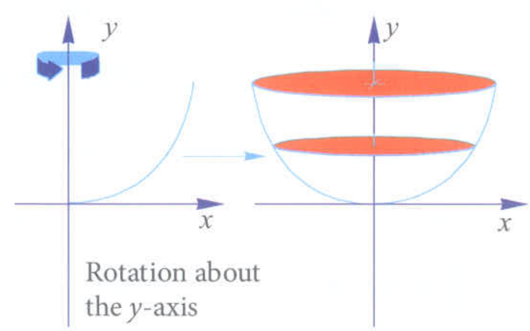
A solid of revolution is formed by revolving a plane region about a line – called the axis of revolution. In this section we will only be using the $x$-axis or the $y$-axis.




For example, in the diagram above, if we revolve the triangular plane region about the vertical axis as shown, we obtain a cone.

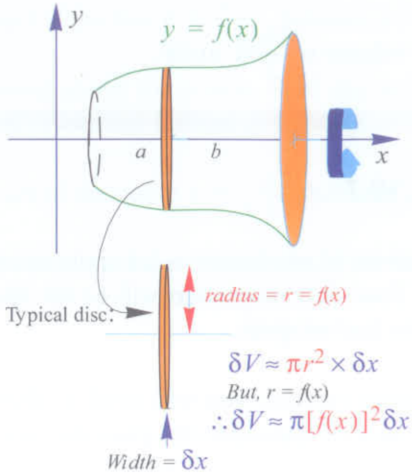
It is important to realize that depending on the axis of revolution, we can obtain very different shapes. For example, if a region bounded by the curve $y = x^2, x \geq 0$ is rotated about the $x$- and $y$- axes, two distinct solid shapes are formed:




When the plane region (enclosed by the curve and the $x$-axis) is rotated about the $x$-axis, the solid object produced is rather like the bell of a trumpet (with a very narrow mouth piece!) or a Malay hat on its side. However, when the plane region (enclosed by the curve and the $y$-axis) is rotated about the $y$-axis, then the solid produced is like a bowl.

Using the same approach as that used when finding the area of a region enclosed by a curve, the $x$-axis and the lines $x = a$ and $x = b$ we have:




Then, the volume, $V$ units³, of such a solid can be cut up into a large number of slices (i.e. discs) each having a width $\delta x$ and radius $f(x_i)$. The volume produced is then the sum of the volumes of these discs, i.e.

$$V = \sum_{i=0}^{i=n-1} \pi\left[f(x_i)\right]^2 \delta x \text{ where } \delta x = \frac{b-a}{n}$$

So, as $n \to \infty, \delta x \to 0$ and so,

$$V = \lim_{\delta x \to 0} \sum_{i=0}^{i=n-1} \pi\left[f(x_i)\right]^2 \delta x = \int_a^b \pi\left[f(x)\right]^2 dx$$

Therefore, we have:

The volume, V units³, of a solid of revolution is given by:

1. when a plane region enclosed by the curve $y = f(x)$ and the lines $x = a$ and $x = b$ is revolved about the $x$-axis.

$$V = \pi \int_{x=a}^{x=b} [f(x)]^2 dx \quad \left[\text{or } V = \pi \int_a^b y^2 dx\right]$$

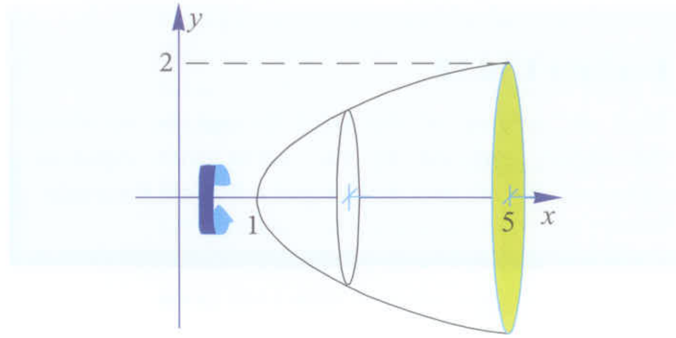
2. when a plane region enclosed by the curve $y = f(x)$ and the lines $y = e$ and $y = f$ is revolved about the $y$-axis.

$$V = \pi \int_{y=e}^{y=f} [f^{-1}(y)]^2 dy \quad \left[\text{or } V = \pi \int_e^f x^2 dy\right]$$

**Example E.10.17**

The curve $y = \sqrt{x-1}, 1 \le x \le 5$ is rotated about the $x$-axis to form a solid of revolution. Sketch this solid and find its volume.

If the same curve is rotated about the $y$-axis, a different solid is formed. Sketch this second solid and find its volume.




The curve has a restricted domain and is rotated about the $x$-axis, so, the solid formed has a volume given by:

$$V = \pi\int_1^5 (\sqrt{x-1})^2 dx = \pi\int_1^5 (x-1)dx$$

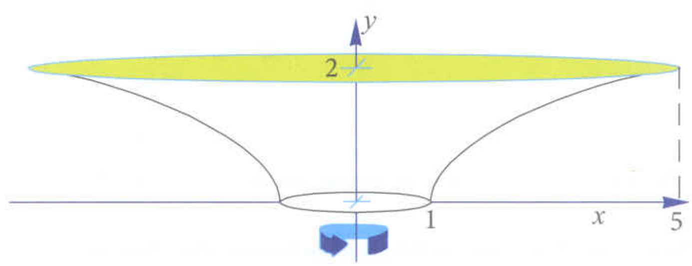
$$= \pi\left[\frac{x^2}{2} - x\right]_1^5$$

$$= \pi\left(\frac{5^2}{2} - 5 - \left(\frac{1^2}{2} - 1\right)\right)$$

$$= 8\pi$$

Therefore, the volume generated is $8\pi$ units³.

If the curve is rotated about the $y$-axis, the solid formed looks like this:




The volume can now be found using the second formula. It is important to realize that the integral limits are in terms of the $y$ variable and so are 0 and 2. Also, $x$ must be made the subject of the rule for the curve:

$$y = \sqrt{x-1} \Rightarrow y^2 = x - 1 \Rightarrow x = y^2 + 1$$

When $x = 1$, $y = 0$ and when $x = 5$, $y = 2$, entering these values into the formula gives:

$$V = \pi\int_0^2 (y^2+1)^2 dy = \pi\int_0^2 (y^4 + 2y^2 + 1)dy =$$

$$\pi\left[\frac{y^5}{5} + \frac{2y^3}{3} + y\right]_0^2$$

$$= \pi\left(\frac{2^5}{5} + \frac{2(2^3)}{3} + 2\right)$$

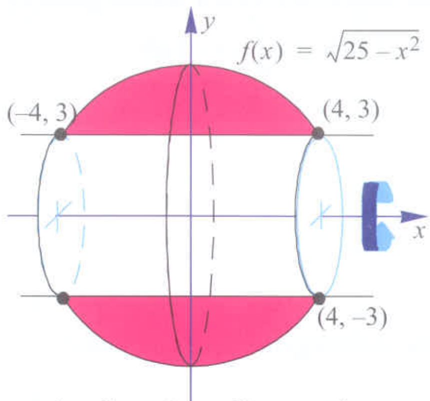
$$= 13\frac{11}{15}\pi$$

i.e. required volume is $13\frac{11}{15}\pi$ units³

**Example E.10.18**

Find the volume of the solid formed by revolving the region enclosed by the curve with equation: $f(x) = \sqrt{25-x^2}$ and the line $g(x) = 3$ about the $x$-axis.

We start by drawing a diagram of this situation. It is a bead.




Next we determine the points of intersection.

Setting $f(x) = g(x)$ we have
$$\sqrt{25-x^2} = 3$$
$$\therefore 25 - x^2 = 9$$
$$\Leftrightarrow x^2 = 16$$
$$\therefore x = \pm 4$$

The solid formed is hollow inside, i.e. from $-3 \le y \le 3$.

Next, we find the difference between the two volumes generated (a little bit like finding the area between two curves):

$$V = V_{f(x)} - V_{g(x)} = \pi \int_{-4}^{4} [f(x)]^2 dx - \pi \int_{-4}^{4} [g(x)]^2 dx$$
$$= \pi \int_{-4}^{4} ([f(x)]^2 - [g(x)]^2) dx$$
$$= 2\pi \int_{0}^{4} ([f(x)]^2 - [g(x)]^2) dx \text{ (by symmetry)}$$
$$= 2\pi \int_{0}^{4} ([\sqrt{25-x^2}]^2 - [3]^2) dx$$
$$= 2\pi \int_{0}^{4} (16 - x^2) dx$$
$$= 2\pi \left[ 16x - \frac{1}{3}x^3 \right]_0^4$$
$$= \frac{256}{3}\pi$$

i.e. required volume is $\frac{256}{3}\pi$ units³.

## Exercise E.10.7

Finding volumes of revolution is an application of definite integration. Your only restriction will be the limitations on your ability to find integrals.

In the following exercise, you will need to draw on all the techniques you have learned in the preceding sections.

* Unless stated otherwise, all answers should be given as an exact value.

1. The part of the line $y = x + 1$ between $x = 0$ and $x = 3$ is rotated about the $x$-axis. Find the volume of this solid of revolution.

2 A curve is defined by $y = \frac{1}{\sqrt{x}}, x \in [1, 5]$. If this curve is rotated about the $x$-axis, find the volume of the solid of revolution formed.

3. The curve $y = \frac{1}{x}$ between the $x$-values $\frac{1}{5}$ and 1 is rotated about the $y$-axis. Find the volume of the solid of revolution formed in this way.

4. Find the volume of the solid of revolution formed by rotating the part of the curve $y = e^x$ between $x = 1$ and $x = 5$ about the $x$-axis.

5. A solid is formed by rotating the curve $y = \sin x, x \in [0, 2\pi]$ about the $x$-axis. Find the volume of this solid.

6. The part of the curve $y = \frac{1}{1-x}$ between the $x$-values 2 and 3 is rotated about the $x$-axis. Find the volume of this solid.

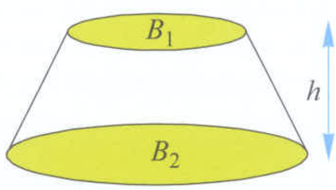
7. The part of the line $y = \frac{x-1}{2}$ between $x = 5$ and $x = 7$ is rotated about the $y$-axis. Find the volume of the solid of revolution formed in this way.

8. The part of the curve $y = \frac{x}{1+x}$ between the $x$-values 0 and 2 is rotated about the $x$-axis. Find the volume of the solid formed in this way.

9. Find the equation of the straight line that passes through the origin and through the point $(h,r)$. Hence use calculus to prove that the volume of a right circular cone with base radius $r$ and height $h$ is given by $V = \frac{1}{3}\pi r^2 h$.

10. Find the equation of a circle of radius $r$. Use calculus to prove that the volume of a sphere is given by the formula $V = \frac{4}{3}\pi r^3$.

11. The diagram shows a shape known as a frustum. Use calculus to prove that its volume is given by the formula




$V = \frac{h}{3}(B_1 + B_2 + \sqrt{B_1 B_2})$ where $B_1$ and $B_2$ are the areas of the circular top and base respectively.

12. The part of the curve $f(x) = \sin\frac{x}{10}$ between $x = 0$ and $x = 5$ is rotated about the $x$-axis. Find the volume of this solid of revolution.

13. The part of the curve $f(x) = x^2 - x + 2$ between $x = 1$ and $x = 2$ is rotated about the $x$-axis. Find the volume of this solid of revolution.

14. a Find the volume generated by the region between the $y$-axis and that part of the parabola $y = x^2$ from $x = 1$ to $x = 3$ when it is rotated about the $y$-axis.

b Find the volume generated by the region between the $x$-axis and that part of the parabola $y = x^2$ from $x = 1$ to $x = 3$ when it is rotated about the $x$-axis.

15. Find the volume of the solid of revolution that is formed by rotating the region bounded by the curves $y = \sqrt{x}$ and $y = \sqrt{x^3}$ about:

a the $y$-axis

b the $x$-axis.

16. Find the volume of the solid of revolution that is formed when the region bounded by the curve with equation $y = 4 - x^2$ and the line $y = 1$ is rotated about:

a the $y$-axis

b the $x$-axis.

17. Find the volume of the solid generated by rotating the region bounded by the curves $y^2 = x^3$ and $y^2 = 2 - x$ about the $x$-axis.

18. The volume of the solid formed when the region bounded by the curve $y = e^x - k$, the $x$-axis and the line $x = \ln 3$ is rotated about the $x$-axis is $\pi \ln 3$ units$^3$. Find $k$.

19. Find the volume of the solid of revolution formed by rotating the region bounded by the axes and the curve $y = \sqrt{3}a\sin x + a\cos x, 0 \le x \le 2\pi$, $a > 0$ about the $x$-axis.

20. If the curve of the function $f(\theta) = \sin k\theta, k > 0, \theta > 0$ is rotated about the $\theta$-axis, a string of sausages is made. Find $k$ such that the volume of each sausage is $\pi$ units$^3$.

21. a On the same set of axes, sketch the curves $y = ax^2$ and:

$$y = 1 - \frac{x^2}{a} \text{ where } a > 0.$$

b Find the volume of the solid of revolution formed when the region enclosed by the curves in part a is:

i rotated about the $y$-axis

ii rotated about the $x$-axis.

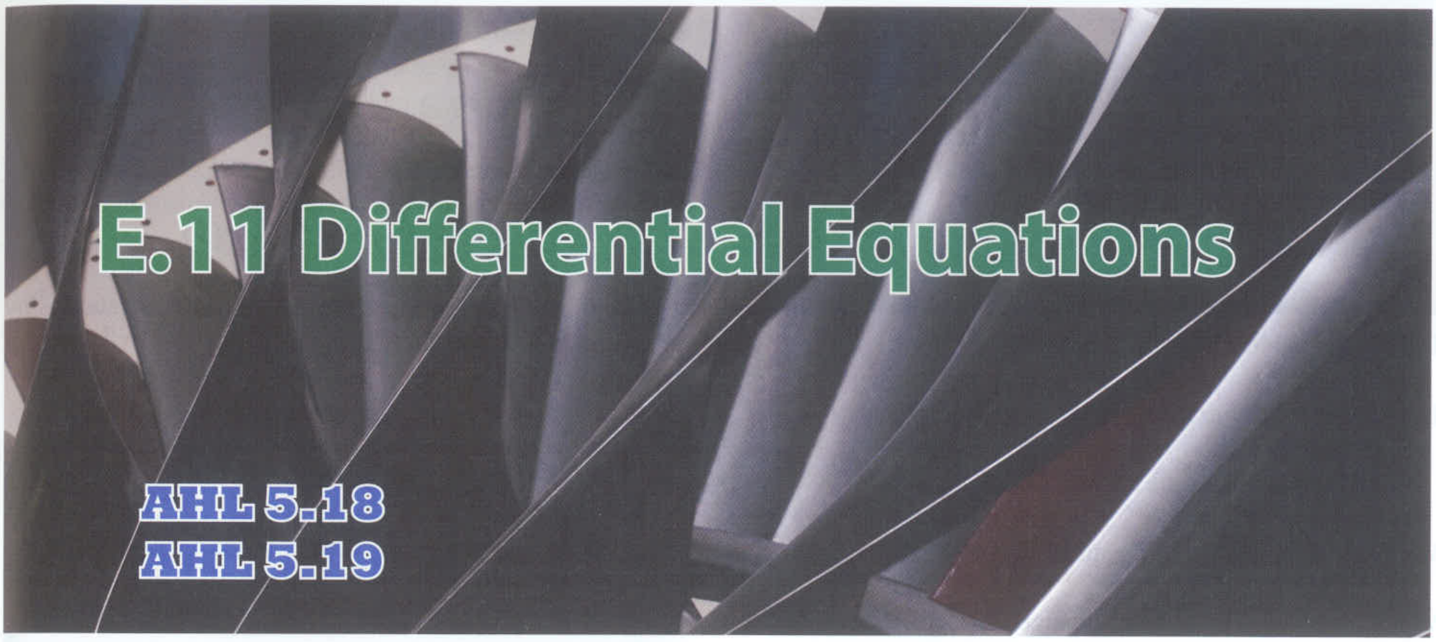
22. On the same set of axes sketch the two sets of points $\{(x, y) : (x-2)^2 + y^2 \le 4\}$ and $\{(x, y) : (x-a)^2 + y^2 \le 4, a \in ]-2,6[\}$.

The intersection of these two sets is rotated about the $x$-axis to generate a solid. Find $a$ if the volume of this solid is $\pi$ units$^3$. Give your answer to three decimal places.

A donut is formed by rotating the curve $\{(x, y) : (x-a)^2 + y^2 = 1, |a| > 1\}$ about the $y$-axis. Find $a$ if the volume of the donut is $100\pi$ units$^3$.

Extra questions

Answers



# E.11 Differential Equations

**AHL 5.18**
**AHL 5.19**

Differential equations deal with situations in which we have information about the rate of change of the variables concerned. This is a wide range of situations and we will look at a few examples to begin with.

### Example E.11.1

Newton's Law of Cooling states that the rate at which a hot body cools is proportional to the difference in temperature between the body and the temperature of the surroundings.

Write a Differential Equation that represents this.

As always, defining variables is a good place to start:

Let: $t$ = time after the start of the measurements

$M$ = temperature of the body at time $t$.

$A$ = ambient temperature (surroundings).

Working from the data in the question:

Newton's Law of Cooling states that the rate at which a hot body cools:

$$\left(\frac{dM}{dt}\right)$$

is proportional to: $\alpha$

the difference in temperature between the body and the temperature of the surroundings:

$$(M - A)$$

This leads to the statement of proportionality:

$$\frac{dM}{dt}\alpha(M-A)$$

and the equation:

$$\frac{dM}{dt}=k(M-A)$$

$k$ is the constant of proportionality.

### Example E.11.2

A function $y=f(x)$ is such that its graph passes through the origin.

The gradient of the graph at the point $(x,y)$ is $2x-4$.

Write a Differential Equation that represents this.

The statement about gradient becomes: $\frac{dy}{dx}=2x-4$

The extra data: $x = 0$, $y = 0$ is sometimes known as an **initial condition**.

**Example E.11.3**

Water leaks out of a tank at a rate that is proportional to its depth.

Write a Differential Equation that represents this.

Defining variables::

Let: $t$ = time after the start of the measurements

$H$ = depth at time $t$.

Working from the data in the question:

Water leaks out - the rate of change of depth with time is negative.

$$\frac{dH}{dt} \alpha H \Rightarrow \frac{dH}{dt} = kH$$

If you are wondering why we have not used '*d*' for depth, it is to avoid the utterly confusing '*dd*' in the derivative.

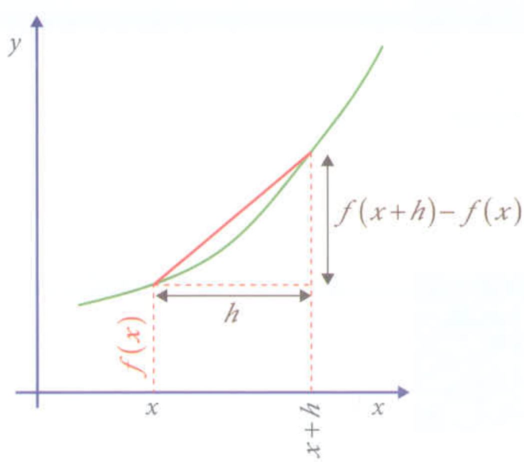
In Example E.11.1 we may seem to have made an eccentric choice of '*M*' rather than '*T*' for temperature. This is because we have already used '*t*' for time. Again, we are trying to avoid confusion.

## Euler's Method

There are many ways of solving differential equations. Euler's Method is a numerical method that depends on using the first principles definition of the derivative:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

A diagram shows this:




Using the slope of the red line:

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}$$

it is possible to infer that:

$$f(x+h) \approx h \times f'(x) + f(x)$$

This is the basis for Euler's Method. We will illustrate this by returning to Example E.11.1.

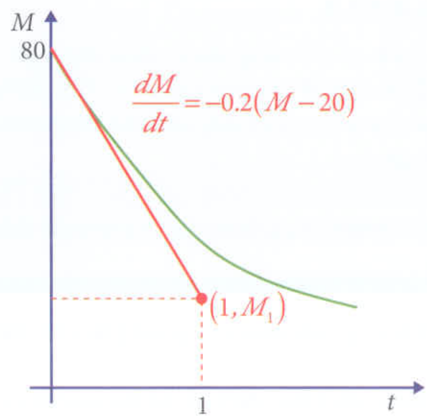
**Example E.11.4**

A cup of tea, initially at 80°C cools at a rate defined by:

$\frac{dM}{dt} = -0.2(M-20)$ where $t$ is measured in minutes.

What is the temperature after 1 minute?

The simplest answer to this question is illustrated in this diagram:




At the start, the tea is cooling at a rate: 
$$\frac{dM}{dt} = -0.2(M-20)$$
$$= -0.2(80-20)$$
$$= -12\ ^\circ/\text{min}$$

This means that, over the first minute, the tea cools by 12° and so will now be 80 – 12 = 68°C.

The approximation here is, however, quite large (the green curve and the red tangent differ in a significant way.

So what can be done to make the approximation better?

As is often the case with calculus, the answer is: take shorter intervals. In this case, we will divide the minute into 10 intervals (each of 6 seconds).

In the first tenth of a minute, the rate of cooling is as before 12 $^\circ\text{min}^{-1}$. However, this lasts for a tenth of a minute and so the cooling is 1.2$^\circ$ and the new temperature is 78.8$^\circ$C.

When we come to the next time interval, we have a new rate of cooling because the tea is at a reduced temperature.

$$\frac{dM}{dt} = -0.2(M-20)$$
$$= -0.2(78.8-20)$$
$$= -11.76 \ ^\circ/\text{min}$$

In the second interval, the tea will cool $0.1\times 11.76 = 1.176^\circ$.

This means that after 0.2 seconds, the tea has cooled to:

78.8 – 1.176 = 77.624 $^\circ$C

For the third time interval, we have a new cooling rate:

$$\frac{dM}{dt} = -0.2(M-20)$$
$$= -0.2(7.624-20)$$
$$= -11.5248 \ ^\circ/\text{min}$$

The actual cooling in the third interval is 1.15248$^\circ$ and the new temperature is 77.624 – 1.15248 = 76.47152$^\circ$.

As this is an approximation, we should not be quoting this level of accuracy. If we continue the process for the ten time intervals, the result (to 3 dec pl) are:

| t | Cooling Rate | Temperature |
|---|---|---|
| 0 | -12.000 | 80.000 |
| 0.1 | -11.760 | 78.800 |
| 0.2 | -11.525 | 77.624 |
| 0.3 | -11.294 | 76.472 |
| 0.4 | -11.068 | 75.342 |
| 0.5 | -10.847 | 74.235 |
| 0.6 | -10.630 | 73.151 |
| 0.7 | -10.418 | 72.088 |
| 0.8 | -10.209 | 71.046 |
| 0.9 | -10.005 | 70.025 |
| 1 | -9.805 | 69.024 |

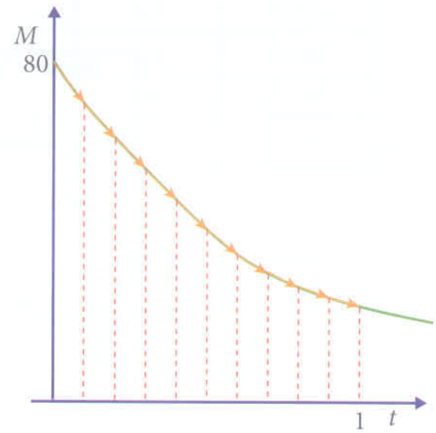
The improved prediction is 69.0$^\circ$C. This is not the same as the single interval result obtained earlier of 68$^\circ$C.

The 'improved' result is larger than the original. This should be as expected since, as the tea cools, the rate of cooling decreases. The more we take this into account, the higher the final temperature will be.

With twice as many intervals the predicted temperature is 69.074$^\circ$C.

This comparatively small alteration in the predicted temperature suggests that we already have an answer good enough to satisfy the requirements of the question.

Graphically, there are ten interval solution 'leapfrogs' down the curve, and that is why we have an improved answer.




The technique of repeating the calculation with narrower intervals in order to assess the accuracy of the answer is a strength of Euler's method. Not that it gives an exact measure of accuracy (such as an interval in which the answer must lie), but it is better than nothing.

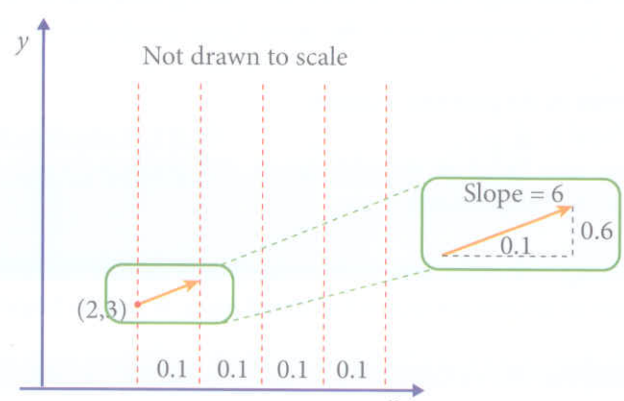
When using numerical methods to solve practical problems, there is almost always a level of accuracy required. Knowing for certain that this has been achieved is an important criterion when it comes to choosing a method.

**Example E.11.5**

A curve is such that $\frac{dy}{dx} = xy,\ y(2) = 3$.

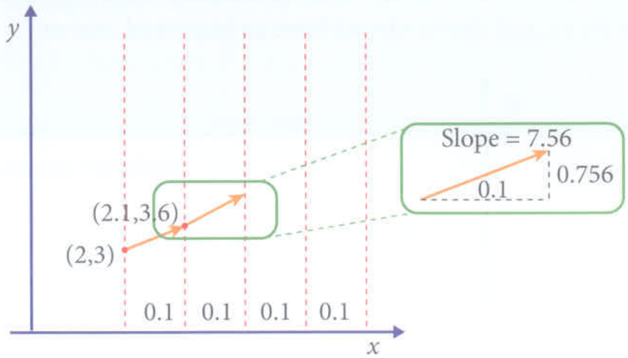
Use Euler's Method with 4 intervals to estimate $y(2.4)$.

At $x = 2, \frac{dy}{dx} = xy = 2\times 3 = 6$



Moving on to the second interval and starting at the point (2.1,3.6).

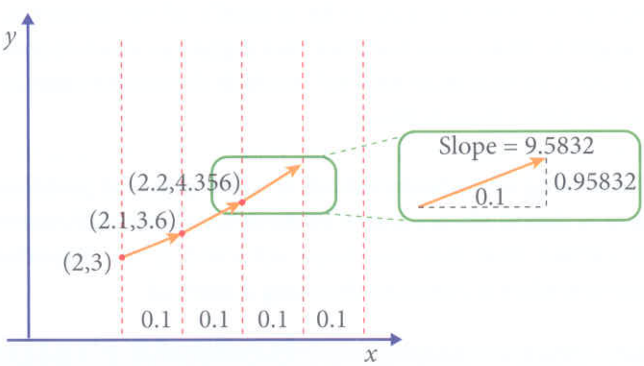
$x = 2.1, \frac{dy}{dx} = xy = 2.1 \times 3.6 = 7.56$




This takes us to the point (2.2,3.6+0.756) = (2.2,4.356).

Proceeding to the next interval.

$x = 2.2, \frac{dy}{dx} = xy = 2.2 \times 4.356 = 9.5832$




This takes us to the point (2.3,4.356+0.95832) = (2.3,5.31432).

Proceeding to the last interval.

$x = 2.3, \frac{dy}{dx} = xy = 2.3 \times 5.31432 = 12.222936$ which gives the solution point as:

(2.3,5.31432+1.2222936) = (2.4,6.5366136).

This is too large a level of accuracy, but by how much? You may be suspecting that the quite large gradients in this example will lead to larger errors in the $y$ values than were evident in the previous example.

If we take twice as many intervals, the predicted value of $y(2.4)$ becomes 6.8508.

You might like to investigate how many intervals are necessary to get an answer correct to 1 decimal place. It is quite large!

**Example E.11.6**

If $p'(t) = -\sqrt{t+1}, p(1) = 4$ use Euler's method to find $p(1.5)$ correct to 1 decimal place.

The first stage begins at the point (1,4).

If we use a single interval, the estimate for $p(1.5)$ is:

$p'(1) = -\sqrt{1+1} \approx -1.414$

The increment in $y$ is $0.5 \times -1.414 \approx -0.707$ so the estimate for $p(1.5)$ is: 4 −0.707 = 3.293.

But how accurate is this?

If we use intervals of 0.1, the results are:

| $t$ | Rate | $p$ |
|---|---|---|
| 1 | -1.414 | 4.00000 |
| 1.1 | -1.449 | 3.85858 |
| 1.2 | -1.483 | 3.71366 |
| 1.3 | -1.517 | 3.56534 |
| 1.4 | -1.549 | 3.41368 |
| 1.5 | -1.581 | 3.25876 |

This gives the estimate $p(1.5)$ = 3.25876

Since we are rounding to 1 decimal place, our current answer is very close the cutoff between rounding up to 3.3 and down to 3.2. A further refinement is indicated.

| $t$ | Rate | $p$ |
|---|---|---|
| 1 | -1.414 | 4.00000 |
| 1.05 | -1.432 | 3.92929 |
| 1.1 | -1.449 | 3.85770 |
| 1.15 | -1.466 | 3.78524 |
| 1.2 | -1.483 | 3.71193 |
| 1.25 | -1.500 | 3.63777 |
| 1.3 | -1.517 | 3.56277 |
| 1.35 | -1.533 | 3.48694 |
| 1.4 | -1.549 | 3.41029 |
| 1.45 | -1.565 | 3.33283 |
| 1.5 | -1.581 | 3.25457 |

This confirms that $p(1.5)$ = 3.3 to 1 dec. pl.

### Exercise E.11.1

1. Estimate the value of $y$ when $x = 2$ given that the curve passes through the origin and:

   $$\frac{dy}{dx} = 2x + 1.$$

   Use Euler's Method and interval sizes of 0.2.

2. Use Euler's Method with five intervals to estimate $f(2)$ if $f(1) = 3$ and:

   $$f'(x) = \frac{1}{x+1}.$$

3. Use Euler's Method with five intervals to estimate $f(1.5)$ if $f(1) = -1$ and:

   $$f'(x) = -\sqrt{x^2+1}.$$

4. Use Euler's Method with five intervals to estimate the value of $y$ when $x = 2$ if the curve passes through (1,1) and:

   $$\frac{dy}{dx} = x^2 y$$

5. Water is leaking from a tank such that the depth $D$ at time $t$ deceases at a rate given by:

   $$D'(t) = -\sqrt{\frac{D}{5}}, D(0) = 5.$$

   Use Euler's Method with five intervals to estimate the depth at time 1.

6. Using Euler's method with a step size of 0.2 calculate an approximate value of $y$ when when $x = 1$ for the differential equation:

   $\frac{dy}{dx} = \frac{x-y}{x+y}$ given that the curve passes through the point (0, 1).

7. In an experiment to determine the effectiveness ($E$) of an enzyme as it varies with pH ($H$), the values of $E'(H)$ were measured:

| $H$ | $E'(H)$ |
|---|---|
| 6 | 0.220 |
| 6.2 | 0.210 |
| 6.4 | 0.140 |
| 6.6 | 0.110 |
| 6.8 | 0.050 |
| 7 | -0.010 |
| 7.2 | -0.060 |
| 7.4 | -0.991 |

   Use Euler's Method to estimate E(7.2) given that E(6) = 3.

Extra questions

## Analytic Solutions

### Separation of Variables

Most analytic methods depend on rearrangements followed by integration. The exact meaning of 'rearrangement' depends on the actual form of the original equation.

Also, most rearrangements allow us to use expressions such as '$dx$' as if they can exist on their own. We have already had some discussion on this when we were considering differentiation from first principles and pointed out that the derivative is a ratio of two infinitesimally small quantities. However, the Leibniz Notation is very useful and does help with these calculations.

**Example E.11.7**
Solve the differential equation $\frac{dy}{dx} = \frac{x}{y}, y(1) = 3.$

This is an example of an equation which can be rearranged so that all the $x$ terms are on one side and all the $y$ terms are on the other.

$$\frac{dy}{dx}=\frac{x}{y}$$

$$x\times dx=y\times dy$$

Both sides can now be integrated with respect to the separate variables.

$$\int x\times dx=\int y\times dy$$

$$\frac{x^2}{2}+c_1=\frac{y^2}{2}+c_2$$

$$\frac{x^2}{2}+c_3=\frac{y^2}{2}$$

$$y^2=x^2+c_4$$

A word on the subject of the constants of integration. We have used four different ones in the above argument. This is not strictly necessary and mostly we would only use one at the end of the calculation.

Next, we use the extra condition $y(1)=3$ to determine the constant:

$$y^2=x^2+c_4$$

$$3^2=1^2+c_4$$

$$c_4=9-1$$

$$=8$$

This gives the solution: $y^2=x^2+8$.

**Example E.11.8**

Solve the differential equation:

$y'(x)=e^{-y}(2x+1), y(1)=1$.

As we observed when solving the previous example, the Leibniz Notation is the most useful in these cases.

$$y'(x)=e^{-y}(2x+1)$$

$$\frac{dy}{dx}=e^{-y}(2x+1)$$

Next, we try to separate the variables:

$$dy=e^{-y}(2x+1)dx$$

$$\frac{1}{e^{-y}}dy=(2x+1)dx$$

$$e^{y}dy=(2x+1)dx$$

That achieved, we can integrate both sides:

$$\int e^{y}\,dy=\int(2x+1)dx$$

$$e^{y}=x^2+x+c$$

We have used only one constant of integration.

Next: $e^{y}=x^2+x+c$

$$y=\log_e\left(x^2+x+c\right)$$

Finally, we use the condition $y(1)=1$ to determine $c$.

$$1=\log_e\left(1^2+1+c\right)$$

$$1=\log_e(2+c)$$

$$e^1=2+c$$

$$c=e-2$$

$$y=\log_e\left(x^2+x+e-2\right)$$

**Example E.11.9**

Solve the differential equation: $\frac{dr}{d\theta}=\frac{\sin\theta}{r}, r(0)=1$.

$$rdr=\sin\theta d\theta$$

$$\int r\,dr=\int\sin\theta\,d\theta$$

$$\frac{r^2}{2}=-\cos\theta+c$$

Since $r(0)=1$:

$$\frac{1^2}{2}=-\cos 0+c$$

$$c=\frac{3}{2}$$

$$\frac{r^2}{2}=-\cos\theta+\frac{3}{2}$$

$$r^2=-2\cos\theta+3$$

$$r=\sqrt{-2\cos\theta+3}$$

What is the domain?

$$-2\cos\theta+3\geq 0$$

$$-2\cos\theta\geq -3$$

$$2\cos\theta\leq 3$$

$$\cos\theta\leq\frac{3}{2}$$

which is all values.

## Exercise E.11.2

1. Solve the differenential equations:

   a $\frac{dy}{dx}=x+1, y(0)=2$

   b $\frac{dy}{dx}=\frac{x+1}{y}, y(1)=2$

   c $\frac{dy}{dx}=\frac{x+1}{1-y}, y(1)=-2$

   d $\frac{dy}{dx}=\frac{1-e^{-2x}}{y}, y(0)=0$

   e $y'(x)=y, y(0)=0$

   f $\frac{dV}{dr}=4\pi r^2, V(0)=0$

   g $y'(x)=\frac{xy^3}{\sqrt{1+x^2}}, y(0)=-1$

   h $\frac{dA}{dr}=\frac{4r^2}{A}, A(0)=0$

   i $\frac{dy}{dx}=\frac{x}{\log_e y}, y(0)=0$

   j $\frac{dy}{dx}=\frac{x-3}{\sqrt{y}}, y(0)=0$

2. The rate of change of the concentration of a chemical in a manufacturing process is proportional to the square root of time (sec). After 100 sec, this rate of change is 0.1 gm sec$^{-1}$. Initially, the concentration is zero.

   a Write a differential equation that represents this situation.

   b Find the solution of this equation.

   c Find the concentration after 50 seconds.

3. A first order chemical reaction depends on the concentration of only one reactant and is directly proportional to that concentration.

   a Write a differential equation that represents this situation.

   b Find the solution of this equation.

   c Find the 'half-life' of the reaction - the time taken for the concentration to halve/double.

4. The stock price of Plastico has risen from \$5.23 to \$5.98 over the past 5 days.

   a Write a differential equation that represents this situation.

   b Is this equation useful in predicting the stock price a year hence?

5. The bubbles released by a scuba diver rise to the surface at an approximately constant velocity. Boyle's Law states that the volume of a fixed mass of gas is inversely proportional to its pressure.

   a Write a differential equation that models the volume of a bubble in terms of time.

   b Find the general solution of that equation.

6. Consider the equation:

   $$\frac{dy}{dt}=e^{y-t}\sec y\left(1+t^2\right), y(0)=0$$

   a Write the equation with separated variables.

   b Find a general (implicit) solution to this equation.

   c Use the initial condition to find the value of the constant of integration.

## Homogeneous Equations

Homogeneous differential equations take the form:

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

They can be solved using the substitution: $y = vx$.

### Example E.11.10

Solve the differential equation: $\frac{dy}{dx} = \frac{x^2 + y^2}{xy}$.

Some rearrangement needs to happen first.

$$\frac{dy}{dx} = \frac{x}{y} + \frac{y}{x}$$
$$= \left(\frac{y}{x}\right)^{-1} + \frac{y}{x}$$

The fact that both terms can be expressed as $\left(\frac{y}{x}\right)$ is crucial to what follows:

Let $y = vx$. It follows that (substituting in the d.e.):

$$y = vx \Rightarrow \frac{y}{x} = v$$
$$\frac{dy}{dx} = \left(\frac{y}{x}\right)^{-1} + \frac{y}{x}$$
$$= v^{-1} + v$$

Also, working from the expression $y = vx$:

$$y = vx$$
$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

by the product rule.

Putting these two results together:

$$v^{-1} + v = v + x\frac{dv}{dx}$$
$$v^{-1} = x\frac{dv}{dx}$$

We can now separate the variables:

$$v^{-1}dx = xdv$$
$$\frac{1}{x}dx = vdv$$

We can now use integration to solve the problem:

$$\int \frac{1}{x}dx = \int v\,dv$$

$$\log_e x + \log_e k = \frac{v^2}{2}$$

Expressing the constant as a logarithm lets us use the Laws of Logs to simplify:

$$\log_e x + \log_e k = \frac{v^2}{2}$$
$$v^2 = 2\log_e kx$$
$$v = \sqrt{2\log_e kx}$$

Finally, we substitute back: $v = \frac{y}{x}$

$$\frac{y}{x} = \sqrt{2\log_e kx}$$
$$y = x\sqrt{2\log_e kx}$$

### Example E.11.11

Solve the differential equation: $x\frac{dy}{dx} = y + xe^{y/x}, y(1) = 2$.

After the experience of the previous example, we are looking to rearrange the equation so we can again use the substitution $y = vx$:

$$x\frac{dy}{dx} = y + xe^{y/x}$$
$$\frac{dy}{dx} = \frac{y}{x} + e^{y/x}$$

As before: $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$$v + x\frac{dv}{dx} = v + e^v$$
$$x\frac{dv}{dx} = e^v$$

Next, separate the variables and integrate.

$e^{-v}dv = \frac{1}{x}dx$

$\int e^{-v}dv = \int \frac{1}{x}dx$

$-e^{-v} = \log_e x + c$

$-e^{-y/x} = \log_e x + c$

Next, use the condition: $y(1) = 2$

$-e^{-2/1} = \log_e 1 + c$

$c = -e^{-2}$

This leads to the (implicit) solution: $-e^{-y/x} = \log_e x - e^{-2}$

## Exercise E.11.3

1. Find the general solution to the homogeneous first order differential equations:

   a $\frac{dy}{dx} = \frac{x+y}{x}$

   b $\frac{dy}{dx} = \frac{y^2+xy}{x^2}$

   c $2xy\frac{dy}{dx} = 3y^2 - x^2$

   d $\frac{dy}{dx} - \frac{y}{x} = e^{y/x}$

   e $\frac{dy}{dx} = \frac{x-y}{x}$

   f $\frac{dy}{dx} = \frac{x-y}{x+y}$

2. Solve: $\frac{dy}{dx} = \frac{3y^2+x^2}{2xy}, y(1) = 2$

3. Find the particular solution for the differential equation:

   $(x^2+y^2)\frac{dy}{dx} = xy$, given that $x = 1$ when $y = 1$.

4. By using the substitution $u = x + y$, show that the differential equation:

   $\frac{dy}{dx} = x + y$

   can be reduced to: $\frac{du}{dx} = x + 1$.

   Hence show that the general solution is given by:

   $x + y + 1 = ke^x$.

5. Use the substitution $u = x + y + 1$, to show that the differential equation:

   $\frac{dy}{dx} = \frac{1}{x+y+1}$ can be reduced to $\frac{du}{dx} - 1 = \frac{1}{u}$.

   Hence show that the general solution is given by:

   $x + y + 2 = ke^y$.

6. Use the substitution $y = ux$, to show that the differential equation:

   $x\frac{dy}{dx} = y^2 + x^2 + y$ can be reduced to $\frac{du}{dx} = u^2 + 1$.

   Hence show that the general solution is given by:

   $y = x\tan(x + c)$.

7. Use the substitution $y = ux$, to show that:

   $\frac{dy}{dx} = \frac{y+\sqrt{x^2+y^2}}{x}, x > 0$ can be reduced to:

   $x\frac{du}{dx} = \sqrt{u^2+1}$

   Hence show that if the curve passes through the point (1, 0), the particular solution is given by:

   $y = \frac{1}{2}(x^2 - 1)$.

8. Solve: $x\frac{dy}{dx}=y+x^2\sin x,\left(\frac{\pi}{2},\frac{\pi}{2}\right)$.

9. Solve: $xy\frac{dy}{dx}=y^2-x^2,(1,0)$

10. Solve: $x\frac{dy}{dx}-2y=x,(1,1)$

## The Integrating Factor

A **first-order linear differential equation**, has the form:

$$\frac{dy}{dx}+P(x)y=Q(x)$$

where $P(x)$ and $Q(x)$ are continuous functions defined on some interval $a<x<b$.

The reason that this type of d.e. is referred to as linear and of the first order is that the dependent variable $y$ and $dy/dx$ both occur in the first degree.

The simplest case of first order d.e.s. occurs when $P(x)=0$ giving us the d.e., $\frac{dy}{dx}=Q(x)$.

If $Q(x)$ is 'nice', this can usually solved by direct integration.

In more complex cases, we may need to use a 'trick' (or rather method) similar to what we needed to do when we used the substitution $y = ux$ in the previous section, the aim of which was to reduce a difficult d.e., into one that is more manageable.

At this stage in your studies, it is probably sufficient to realise that these methods work. If you have time, it is always a good idea to understand why they work.

In this section, we will look at multiplying $\frac{dy}{dx}+P(x)y=Q(x)$ throughout by: $e^{\int P(x)dx}$.

### Example E.11.12

Find the general solution to the differential equation:

$$\frac{dy}{dx}-\frac{2}{x}y=x^3$$

Comparing $\frac{dy}{dx}-\frac{2}{x}y=x^3$ with the general form, $\frac{dy}{dx}+P(x)y=Q(x)$ we have: $P(x)=-\frac{2}{x},Q(x)=x^3$.

The integrating factor is: $e^{\int P(x)dx}=e^{\int -\frac{2}{x}dx}=e^{-2\log_e x}=x^{-2}(+c)$

We will drop the constant at this stage, but it cannot be forgotten entirely. We are after one integrating factor, not the general case.

The method now depends on multiplying the d.e. throughout by the integrating factor to get:

$$\left(\frac{dy}{dx}-\frac{2}{x}y\right)\times x^{-2}=\left(x^3\right)\times x^{-2}$$

This does not look any better until we realise that the left hand side can be written as a single derivative. This is because of the Product Rule of differentiation:

$$\frac{d}{dx}\left(y\times x^{-2}\right)=\frac{dy}{dx}\times x^{-2}+(-2)x^{-3}\times y$$
$$=\left(\frac{dy}{dx}-\frac{2}{x}y\right)\times x^{-2}$$

This is the same as the left hand side of the d.e. This can now be written:

$$\frac{d}{dx}\left(y\times x^{-2}\right)=x^3\times x^{-2}$$
$$\frac{d}{dx}\left(y\times x^{-2}\right)=x$$

Next, integrate both sides with respect to $x$:

$$y\times x^{-2}=\int x\,dx$$
$$y\times x^{-2}=\frac{x^2}{2}+c$$

Recall that integration and differentiation are inverse processes and undo one another. Note also that the constant of integration now plays an important role in the solution.

$$y\times x^{-2}=\frac{x^2}{2}+c$$
$$y=\frac{x^4}{2}+cx^2$$

This is the general solution of the d.e.

**Example E.11.13**

Find the general solution to the differential equation:

$y'+2y=2e^x$.

The first step is to compare the question with the standard d.e.

$$y'+2y=2e^x$$

$$\frac{dy}{dx}+P(x)y=Q(x)$$

The two will match if we have $P(x)=2, Q(x)=2e^x$.

The integrating factor is: $e^{\int P(x)dx}=e^{\int 2dx}$
$=e^{2x}$

Multiplying both sides by this factor gives:

$$(y'+2y)\times e^{2x}=2e^x\times e^{2x}$$
$$=2e^{3x}$$

As in the previous example, the left hand side can be written as a single derivative. Use the Product Rule to check that this is so.

$$\frac{d}{dx}\left(y\times e^{2x}\right)=2e^{3x}$$

Next, integrate both sides.

$$y\times e^{2x}=\int 2e^{3x}\,dx$$
$$=\frac{2}{3}e^{3x}+c$$
$$y=\frac{2}{3}e^x+ce^{-2x}$$

Before proceeding, we will check that this is the correct solution. This is good general practice, but you will seldom have the time to do it in an examination. The check begins by working from the solution and differentiating.

$$y=\frac{2}{3}e^x+ce^{-2x}$$
$$y'=\frac{2}{3}e^x-2ce^{-2x}$$

Substituting in the d.e. $\frac{2}{3}e^x-2ce^{-2x}+2\left(\frac{2}{3}e^x+ce^{-2x}\right)=2e^x$

$$\frac{2}{3}e^x+\frac{4}{3}e^x-2ce^{-2x}+2ce^{-2x}=2e^x$$

which is true, confirming the answer.

**Example E.11.14**

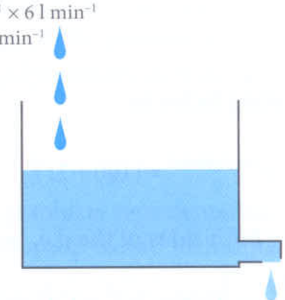
A container is initially filled with 100 litres of a salt solution containing 50 kg of salt. Brine containing 2 kg of salt per litre runs into a tank at a rate of 6 litres/min, and the mixture runs out at a rate of 4 litres/min.

a Assuming that the mixture is kept uniform by stirring, set up a differential equation describing the relationship for the amount of salt in the container at any time $t$ minutes.

b How much salt will there be in the tank after 10 minutes?

a Let $s(t)$ kg be the amount of salt in the container at any time $t$ minutes.

Amount in $= 2\text{ kg l}^{-1}\times 6\text{ l min}^{-1}$
$= 12\text{ kg min}^{-1}$



Amount out $=\left(\frac{s}{100+2t}\text{kg l}^{-1}\right)\times\left(4\text{ l min}^{-1}\right)$
$=\frac{4s}{100+2t}\text{kg min}^{-1}$

Then, we have that the rate of change in the amount of salt in the container must be given by:

$$\frac{ds}{dt}=(\text{Amount in} - \text{Amount out})\ (\text{per minute})$$

The 'Amount in' (per minute) is 12 kg min$^{-1}$, while the 'Amount out' per minute is $=\frac{4s}{100+2t}$ kg min$^{-1}$.

The last term was derived as follows:

First, we need to determine the concentration, $C$, of salt in the container at any time $t$, and, given:

$$C=\frac{\text{Amount of salt in container at any time } t}{\text{Volume of solution in container at any time } t}$$
$$=\frac{s}{100+(6-4)t}$$

this is because:

i by definition, we have that $s$ kg is the amount of salt in the tank at any time $t$,

ii initially there are 100 litres of solution in the container and every minute this increases by $(6 - 4) = 2$ litres, so that after $t$ minutes, there will be an extra $2t$ litres in the container.

Therefore, we have the differential equation:

$$\frac{ds}{dt}=12-\frac{4s}{100+2t}, t\geq 0$$

Rearranging this so that it becomes directly comparable with the standard form:

$$\frac{ds}{dt}+\frac{4s}{100+2t}=12$$

Therefore: $P(t)=\frac{4}{100+2t}, Q(t)=12$

The integrating factor is:

$$e^{\int\frac{4}{100+2t}dt}=e^{2\log_e(100+2t)}=(100+2t)^2$$

Multiplying both sides of the d.e. by this:

$$\left(\frac{ds}{dt}+\frac{4s}{100+2t}\right)(100+2t)^2=12\left((100+2t)^2\right)$$

As before, the left had side resolves to the single derivative:

$$\frac{d}{dt}(s)(100+2t)^2=12\left((100+2t)^2\right)$$

Integrating:

$$s(100+2t)^2=\int 12\left((100+2t)^2\right)dt$$
$$=2(100+2t)^3+c$$
$$s=2(100+2t)+\frac{c}{(100+2t)^2}$$

Now, when $t = 0$, $s = 50$ so $c = -1\ 500\ 000$ and:

$$s=2(100+2t)+\frac{1500000}{(100+2t)^2}, t\geq 0$$

Therefore, when $t = 10$, $s = 135.83$, i.e., approximately 135.83 kg of salt remains.

## Exercise E.11.4

1. Find the general solution to the following first order linear differential equations:

a $\frac{dy}{dx}-\frac{2}{x}y=x^4, x>0$

b $\frac{dy}{dx}+\frac{2}{x}y=x^4, x>0$

c $\frac{dy}{dx}+2y=e^{-x}$

d $\frac{dy}{dx}+2y=x$

e $\frac{dy}{dx}+\frac{1}{x}y=\frac{\ln x}{x^2}, x>0$

f $\frac{dy}{dx}+\frac{y}{x-1}=x$

2. Solve the following differential equations:

a $\frac{dy}{dx}=e^x-y, y(0)=1$

b $(x^2+1)\frac{dy}{dx}-xy=0, y(0)=3$

c $5\frac{dx}{dt}+\frac{15x}{50-t}=1, x(0)=-45$

d $x\frac{dy}{dx}+y=4x^2, y(1)=0$

e $\frac{dy}{dx}=\frac{1}{x+y^2}, y(-1)=0$

Extra Questions

# Series

A familiar series is the Geometric Series (GS) which is generated by a starting term, $a$ (where $a \in \mathbb{R}$), and a ratio, $r$ (where $r \in \mathbb{R}$), to give each term equal to $r$ times the previous term and the series is the sum of these terms. The sequence of the first $n$ terms is:

$a_1, a_2, a_3, \ldots a_{n-1}, a_n$ which is $a, ar, ar^2, \ldots ar^{n-2}, ar^{n-1}$.

and the sum of these terms is the $n$th partial sum,

$$S_n = \sum_{i=1}^{n} a_i . r^{i-1}$$

This is: $S_n = a + ar + ar^2 + \ldots + ar^{n-2} + ar^{n-1}$

$$= \frac{a(1-r^n)}{1-r}$$

This formula was derived in Chapter A2 of the Core text.

For an infinite number of terms, if $|r|<1$, then the geometric series converges (in the limit as $n \to \infty$. Why?) to the sum:

$$a + ar + ar^2 + \ldots. = \frac{a}{1-r}$$

We set $a = 1$ (since $a$ is just a multiplicative constant) and $r = -x$ and so the sum of the geometric series becomes the formula:

$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots$ that is: $\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n . x^n$

which is valid for $|x|<1$. This geometric series is now used as a formula to expand a function, $1/(1+x)$, as a series with terms of powers of the variable $x$ (with real coefficients). This is an example of a power series. In applications, the infinite series is truncated giving a polynomial approximation, of degree $n$, which is an approximation of the function.

The function $f(x) = \frac{1}{1+x}$ is approximated by a series of polynomials.

For example, $P_0 = 1$

$P_1 = 1 - x$

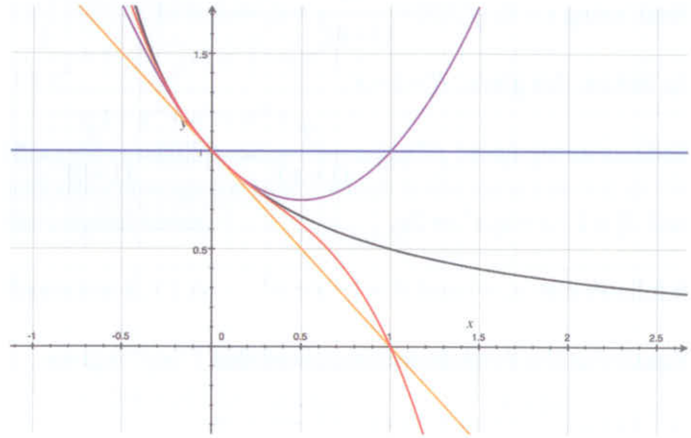
$P_2 = 1 - x + x^2$

$P_3 = 1 - x + x^2 - x^3$

Whilst none of these polynomial are the same as $f(x) = \frac{1}{1+x}$ they do 'approximate' it.

The approximation is centred on the point (0,1). $P_0$ has the right intercept. $P_1$ has the right intercept and the right slope at (0,1). $P_3$ also starts to 'bend' in the same way as $f$ etc.

Graphically, this is:




The colour coding is:

$f$ is black, the constant function $P_0$ is blue, the linear function is orange, the quadratic is purple and the cubic is red.

One very important thing to notice is that the extent to which the polynomials match to $f$ spreads out from (0,1) as we take higher powers of the polynomials. This is because the derivatives of the polynomials increasingly match those of $f$.

### Exercise E.11.5

1. Extend the graphical representation of the polynomial approximations up to and including the fifth power.

2. If we use $P_5$ to estimate the value of $\frac{1}{1.1}$, what is the result and what error is made?

3. Prove that $\frac{1}{1.1} = 0.\dot{9}\dot{0}$.

Formally, the 'best' polynomial approximation (of degree $n$) of a function, $f(x)$, means that the function and the polynomial have the same value at $x = 0$, and all derivatives (up to and including the $n$th derivative) are the same at $x = 0$ for the function and for the polynomial approximation.

Returning to: $f(x) = \frac{1}{1+x}$ and our objective of finding a polynomial of the form: $a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots$ that is equal to it.

We have: $f(0) = \frac{1}{1+0} = 1$ so $P_0 = a_0 = 1$ (as before).

Next, we look at the first derivatives of both $f$ and $P_1$.

$f(x) = \frac{1}{1+x} \Rightarrow f'(x) = \frac{-1}{(1+x)^2}$ - chain or quotient rule.

$P_1(x) = 1 + a_1 x \Rightarrow P_1'(x) = a_1$

Next, using $x=0$: $f'(0)=\dfrac{-1}{(1+0)^2}=a_1 \Rightarrow a_1=-1$.

As before, this gives: $P_1=1-x$.

Differentiating again: $f''(x)=\dfrac{2}{(1+x)^3} \Rightarrow f''(0)=\dfrac{2}{(1+0)^3}=2$

and $P_2=1-x+a_2x^2 \Rightarrow 2a_2$

It follows that $a_2=1$ and $P_2=1-x+x^2$

Equality of the third derivatives (recall that $f'''=f^{(3)}$) gives:

$$f^{(3)}(x)=\frac{-2\times 3}{(1+x)^4} \Rightarrow f^{(3)}(x)=\frac{-2\times 3}{(1+0)^4}=-6$$

and for: $P_3=1-x+x^2+a_3x^3$ so that $P_3^{(3)}=3\times 2a_3$.

This gives $P_3=1-x+x^2-x^3$.

The pattern continues. For the example here, obtained from the geometric series, we obtain the same power series by progressively finding the 'best' polynomial approximations: this is called the Maclaurin Series for the function, named for the Scottish mathematician Colin Maclaurin (1698 – 1746).

The advantage of this calculus based method is that it can be applied to a wide range of functions as our next example shows.

**Example E.11.15**

Find the first 3 non-zero terms of the Maclaurin series for sin*x*.

We assume that the function $f(x)=\sin x$ can be expressed as a polynomial $P(x)=a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+a_5x^5+\ldots$

All our calculations will occur at $x=0$.

$f(0)=\sin 0=0, P(0)=a_0$ so we now have:

$f(x)=\sin(x) \Rightarrow f(0)=\sin(0)=0 \Rightarrow a_0=0$

Next, we differentiate both the approximating polynomial and the function:

$$P'(x)=a_1+2a_2x+3a_3x^2+4a_4x^3+5a_5x^4+\ldots \Rightarrow P'(0)=a_1$$
$$f'(x)=\cos(x) \Rightarrow f'(0)=\cos(0)=1 \Rightarrow a_1=1$$

This gives the first non-zero coefficient.

Differentiate again:

$$P''(x)=2a_2+3.2a_3x+4.3a_4x^2+5.4a_5x^3+\ldots \Rightarrow P''(0)=2a_2$$
$$f''(x)=-\sin(x) \Rightarrow f''(0)=-\sin(0)=0 \Rightarrow a_2=0$$

Differentiate again:

$$P^{(3)}(x)=3.2.1a_3+4.3.2a_4x+5.4.3a_5x^2+\ldots \Rightarrow P^{(3)}(0)=3.2.1a_3$$
$$f^{(3)}(x)=-\cos(x) \Rightarrow f^{(3)}(0)=-\cos(0)=-1 \Rightarrow a_3=\frac{-1}{3.2.1}$$

and again:

$$P^{(4)}(x)=4.3.2.1a_4+5.4.3.2a_5x+\ldots \Rightarrow P^{(4)}(0)=4.3.2.1a_4$$
$$f^{(4)}(x)=\sin(x) \Rightarrow f^{(4)}(0)=\sin(0)=0 \Rightarrow a_4=0$$

and again:

$$P^{(5)}(x)=5.4.3.2.1a_5+\ldots \Rightarrow P^{(5)}(0)=5.4.3.2.1a_5$$
$$f^{(5)}(x)=\cos(x) \Rightarrow f^{(5)}(0)=\cos(0)=1 \Rightarrow a_5=\frac{1}{5.4.3.2.1}$$

So far, we have: $f(x)=\sin(x)\approx x-\dfrac{x^3}{3.2.1}+\dfrac{x^5}{5.4.3.2.1}$

or, using the factorial notation:

$$f(x)=\sin(x)\approx x-\frac{x^3}{3!}+\frac{x^5}{5!}$$

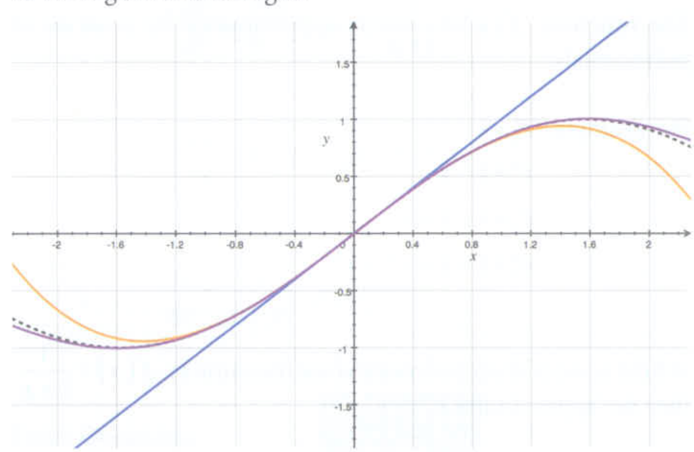
This is a pattern that continues indefinitely so that, using the sigma notation:

$$\sin(x)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

This is a complex formula, so look at the parts that make it up. We only have odd powers. That is why we have $2n+1$. Substitute $n=0, 1, 2,\ldots$ and you get the odd numbers 1, 3, 5,...

The $(-1)^n$ gives the alternating signs.

As we did before, we will look at the approximating polynomials and the extent to which they fit the sine function in the region of the origin.




Sine is dotted, linear blue, cubic orange and quintic purple.

## Convergence

We have discussed the idea of convergence in the context of geometric series.

$\frac{1}{1+x}=1-x+x^2-x^3+...$ converges only for $-1 < x < 1$.

This is due to the properties of powers of $x$. If $x$ is bigger than 1 or less than −1, these are large. Between these two, powers of $x$ get small.

The series $\sin(x)\approx x-\frac{x^3}{3!}+\frac{x^5}{5!}+...$ is converges for all real values of $x$.

This is despite the powers of $x$ in the numerator. The factorials in the denominators 'outgrow' them and the terms decrease rapidly.

Note. The degree 1 polynomial approximation $\sin(x)\approx x$ has at least two uses:

**Physics:**

$\sin(x)\approx x$ is an approximation used in physics to model the behaviour of an oscillating pendulum. The differential equation obtained has a sin($x$) term and it is too difficult to solve (exactly). However if $\sin(x)\approx x$ is used, the differential equation is easily solved. This is a reasonable approximation for small values of $x$ (narrow pendulum swings).

**Navigation:**

A pilot who observes a cross-wind correction of 5° (the difference between the actual track and the aircraft heading - quite easy to observe) has to know only that 1° ≈ 0.02 radians. The rest can be done in the head. 5° ≈ 0.1 radians. If the airspeed is 200 knots (hypotenuse) the cross-wind is 20 knots.

## Other examples

The differentiation method can be used in a number of cases (see the next exercise).

So can direct substitution, as our next example shows.

**Example E.11.16**

Find a series expansion for: $\frac{1}{1+x^2}$

We use the first series in this section $\frac{1}{1+x}=1-x+x^2-x^3+...$

We can achieve our result by substituting $x^2$ for $x$.

$$\frac{1}{1+x^2}=1-\left(x^2\right)+\left(x^2\right)^2-\left(x^2\right)^3+...$$
$$=1-x^2+x^4-x^6+...$$

Since the series expansion converges for $|x| < 1$, the new series will also converge for $|x| < 1$ which is the same interval as for the original series.

## Exercise E.11.6

1. Use the differentiation method to find series expansions for:

   a $f(x)=\cos(x)$

   b $f(x)=e^x$

   c $f(x)=\log_e(1+x)$

2. Use the substitution method to find series for:

   a $f(x)=e^{-x}$

   b $f(x)=\log_e(1-x)$

3. Prove that: $\frac{1}{(1-x)^2}=\sum_{n=0}^{\infty}nx^{n-1}$

4. Use the identity: $\tan(x)=\frac{\sin(x)}{\cos(x)}$ to find the first three terms in the expansion of the MacLaurin series for $\tan(x)$.

5. Use the GS expansion for $\frac{1}{1+x}$ to write down the Maclaurin series for $f(x)=\frac{2}{3-x}$.

   a for what $x$ values does the series converge?

   b Show the $f(x)$ and the cubic approximation on the same plot.

6. Use a series argument to prove that:

   $e^{ix}=\cos(x)+i\sin(x)$

# Maclaurin Series and Differential Equations

In previous sections, we have solved differential equations of the form:

$$\frac{dN}{dt} = -\lambda N$$

mainly in the context of natural decay such as radioactivity.

The differential equation's solution, with $C$ an arbitrary constant, is:

$$N = e^{-\lambda t + C} = C.e^{-\lambda t}$$

For radioactive decay, the initial number of atoms (at time zero), $N_0 = N(0)$, is known. Thus the solution of the initial value problem is:

$$N = N_0.e^{-\lambda t}$$

Alternatively, we now obtain this solution by directly computing the Maclaurin series. For this problem, we notice that we can obtain the full Maclaurin series and recognize the solution function (which is not generally possible).

Let the solution $N(t)$ have a Maclaurin series

$$N(t) = \sum_{i=0}^{\infty} a_i t^i \left[ = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \dots \right]$$

The derivative, $N'(t)$, is obtained by term by term differentiation (with respect to $t$). The differential equation becomes:

$$\begin{aligned} N'(t) &= a_1 + 2a_2 t + 3a_3 t^2 + 4a_4 t^3 + \dots \\ &= -\lambda\left(a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \dots\right) \end{aligned}$$

This is true for all valid values of $t$, so the coefficients of any given power of $t$ must be the same for both sides of the equation. Equate coefficients as follows:

$$t^0 : a_1 = -\lambda a_0$$

$$t^1 : a_2 = \frac{-\lambda a_1}{2} = \frac{(-\lambda)^2 a_0}{2}$$

$$t^2 : a_3 = \frac{-\lambda a_2}{3} = \frac{(-\lambda)^3 a_0}{3!}$$

$$t^3 : a_4 = \frac{-\lambda a_3}{4} = \frac{(-\lambda)^4 a_0}{4!}$$

etc.

The quartic Maclaurin polynomial is:

$$\begin{aligned} p_4 &= a_0 - a_0\lambda t + a_0\frac{\lambda^2}{2!}t^2 - a_0\frac{\lambda^3}{3!}t^3 + a_0\frac{\lambda^4}{4!}t^4 \\ &= a_0\left(1 - \lambda t + \frac{\lambda^2}{2!}t^2 - \frac{\lambda^3}{3!}t^3 + \frac{\lambda^4}{4!}t^4\right) \end{aligned}$$

The pattern continues. Compare this with the series you found for the exponential function in Exercise E.11.6 question 1.b. It follows that the Maclaurin series solution of the differential equation is:

$N(t) = a_0.e^{-\lambda t}$ where is $a_0$ an arbitrary value.

If we are talking about radioactive decay, $a_0$ is the initial number of atoms and we have the solution already discussed.

$N(t) = N_0.e^{-\lambda t}$.

**Example E.11.17**

Solve: $y' = -2xy$

Begin by assuming that a series solution exists and that:

$$y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \dots$$

Differentiate term by term:

$$y'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \dots$$

Next, substitute these two series into both sides of the differential equation:

Left: $y'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \dots$

Right: $-2xy = -2x\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \dots\right)$

$$= -2a_0 x - 2a_1 x^2 - 2a_2 x^3 - 2a_3 x^4 \dots$$

If these two are to be identical, then they must be the same for each power of $x$ in turn (this is called 'equating coefficients').

$$x^0 : a_1 = 0$$

$$x^1 : 2a_2 = -2a_0 \Rightarrow a_2 = -a_0$$

$$x^2 : 3a_3 = -2a_1 \Rightarrow a_3 = 0$$

$$x^3 : 4a_4 = -2a_2 = 2a_0 \Rightarrow a_4 = \frac{a_0}{2}$$

So far we have generated the series:

$$y(x)=a_0-a_0x^2+\frac{a_0}{2}x^4+...$$

This is not really far enough to establish a pattern. For example, is the next term going to be negative? Is the next denominator going to be 4 or 4! or something else?

$$y'(x)=...+3a_3x^2+4a_4x^3+5a_5x^4+6a_6x^5+...$$
$$-2xy=...-2a_1x^2-2a_2x^3-2a_3x^4-2a_4x^5-2a_5x^6-...$$

Continuing to equate coefficients:

$$x^4:5a_5=-2a_3\Rightarrow a_5=0$$

$$x^5:6a_6=-2a_4=2a_0\Rightarrow a_6=-\frac{a_0}{3\times 2}$$

We can now add the next term to our series:

$$y(x)=a_0-a_0x^2+\frac{a_0}{2}x^4-\frac{a_0}{3\times 2}x^6...$$

The pattern is much clearer now, though you may wish to clarify it by continuing this process. The solution series appears to be:

$$y(x)=a_0-a_0x^2+\frac{a_0}{2}x^4-\frac{a_0}{3!}x^6+\frac{a_0}{4!}x^8...$$
$$=a_0\left(1-x^2+\frac{1}{2!}x^4-\frac{1}{3!}x^6+\frac{1}{4!}x^8...\right)$$

We can, however go a step further by identifying the series as

$$y(x)=a_0e^{-x^2}$$

Our next example is related to the previous one and we will solve it using the sigma notation.

**Example E.11.18**

Solve: $y'=2x(1+y)$

Let $y$ have the Maclaurin series: $y(x)=\sum_{i=0}^{\infty}a_ix^i$.

As before the derivative of the left hand side of the differential equation is obtained by term by term differentiation. The differential equation becomes:

$$\sum_{i=0}^{\infty}i.a_ix^{i-1}=2x(1+y)$$
$$=2x+2x.\sum_{i=0}^{\infty}a_ix^i$$
$$=2x+2\sum_{i=0}^{\infty}a_ix^{i+1}$$

The key step is now to equate coefficients of powers of $x$. The summations have different looking powers of $x$, so to make it easy, we re-label the powers of $x$ in the summations. On the left hand side, let $j=i-1$; the summation starts from $i=1$, so $j=0$.

Similarly, on the right hand side, let $k=i+1$. The summation starts from $i=0$ so $k=1$, thus:

$$\sum_{j=0}^{\infty}(j+1).a_{j+1}x^j=2x+2\sum_{k=1}^{\infty}a_{k-1}x^k$$

The next step is to equate coefficients of powers of $x$ (with $j=k$), as follows:

$$x^0:a_1=0+0$$
$$x^1:2a_2=2+2a_0\Rightarrow a_2=1+a_0$$
$$x^2:3a_3=+2a_1=0$$
$$x^3:4a_4=+2a_2\Rightarrow a_4=\frac{1+a_0}{2}$$
$$x^4:5a_5=+2a_3=0$$
$$x^5:6a_6=+2a_4\Rightarrow a_6=\frac{1+a_0}{3!}$$
$$x^6:7a_7=+2a_5=0$$
$$x^7:8a_8=+2a_6\Rightarrow a_8=\frac{1+a_0}{4!}$$

etc.

Note that all odd powers (in the Maclaurin series for $y$) have coefficient zero.

Note also that is arbitrary. Since $1+a_0$ is repeated in the coefficients as a multiplicative constant, we simplify the expression by setting $K=1+a_0$, so $a_0=K-1$ where $K$ becomes the arbitrary constant.

The degree 8 Maclaurin polynomial for the solution of the differential equation is:

$$p_8=K-1+Kx^2+\frac{K}{2!}x^4+\frac{K}{3!}x^6+\frac{K}{4!}x^8$$
$$=K\left(1+x^2+\frac{1}{2!}x^4+\frac{1}{3!}x^6+\frac{1}{4!}x^8\right)-1$$

Now compare this with the series you would obtain by making the substitution $x\to x^2$ in the series for the exponential function.

The solution for $y'=2x(1+y)$ is:

$$y(x)=K\sum_{n=0}^{\infty}\frac{1}{n!}(x^2)^n-1=Ke^{x^2}-1$$

If you have found the use of the sigma notation confusing, make sure you work through the first question in the next exercise.

Solving differential equations are closely related problems.

Series can help with some 'difficult' integrals as our final example should demonstrate.

**Example E.11.19**

Evaluate: $\int_0^1 \frac{\sin(x)}{x} dx$

There is an interesting question here as to whether or not this integral even exists as we seem to have a zero denominator at the lower limit. However, a quick visit to L'Hôpital's Rule should convince you that all is well.

The series for the sine function is:

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

It follows that: $\frac{\sin(x)}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \ldots$

Since this is polynomial, it can be integrated term by term.

$$\int_0^1 \frac{\sin(x)}{x} dx = \int_0^1 \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \ldots\right) dx$$

$$= \left[x - \frac{x^3}{3\times 3!} + \frac{x^5}{5\times 5!} - \frac{x^7}{7\times 7!} + \ldots\right]_0^1$$

$$= 1 - \frac{1}{18} + \frac{1}{600} - \frac{1}{35280} + \ldots$$

Is this far enough to be sure we have 5 significant figures?

The decimal answer so far is: 0.9460827664

The next term is $\frac{1}{9\times 9!} \approx 3\times 10^{-7}$. We have enough accuracy.

The integral is: 0.94608 to 5 s.f.

## Exercise E.11.7

1. Consider the problem of Example E.11.17:

   Solve: $y' = 2x(1+y)$

   Rework this example without using the sigma notation.

2. For the initial value problem: $y'(x) = 2 - y, y(0) = 1$

   a Find the exact solution.

   b Find the exact Maclaurin polynomial of degree 4.

   c Show the answers to parts a & b graphically.

3. For the initial value problem: $y'(x) = 2 + 2y, y(0) = 1$

   a Find the exact solution.

   b Find the exact Maclaurin polynomial of degree 4.

   c Show the answers to parts a & b graphically.

4. Find, correct to 5 significant figures: $\int_0^1 x^2.e^{-2x} dx$

5. Find, correct to 5 significant figures $\int_0^1 x^2.\sin(x) dx$

Extra questions

Answers

# Index

## A

Absolute Value Function 93
Algebra of complex numbers 24
Analytic Solution 251
Antoine, Guillaume François 209
Argand Diagram 29
Axial intercept 78

## B

base vector notation 130
Bayes' Theorem 180
Binomial Theorem 10

## C

Cardano, Girolamo 23, 74
Cartesian Equation of a Plane 166
Cartesian Representation of Vectors 129
Combination 6
Complex number 23
Complex Number 24
Component form 160
Compound Angle Identities 115
Conjugate Root Theorem (C.R.T) 43
Continuity 200
Continuous Random Variable 189
Contradiction 47
Convergence 261
Crosswind Landing 136

## D

de Moivre's Theorem 35
Differentiability 203
Differential Equation 247, 262
Dilation 197
Discrete Distribution 185
Discrete domain 200

## E

Elimination method 58
Equal vectors 126
Euler Form 34
Euler's Method 248
Even function 85

## F

Factorising 67
Factor Theorem 43, 66
Fiore, Antonio 73
Fractal 46
Fundamental Theorem of Algebra 43, 66

## G

Geometric Proof 134
Geometric Series 259

## H

Higher Derivatives 205
Homogeneous Equation 254
Horizontal asymptote 77

## I

Identity and inverse functions 86
Imaginary Number 64
Implicit Differentiation 213
Implicit relation 213
Induction 49
Inequalities 95
Inequations 95
Integration 225
Integration by Parts 239
Intersection of a Line and a Plane 172
Intersection of Three Planes 175
Intersection of Two Lines 172
Intersection of Two Planes 174
Inverse Cosine Function 110

Inverse functions 86
Inverse Sine Function 109
Inverse Tangent Function 112

J

Joukowsky aerofoil 92
Joukowsky transformation 92
Jump discontinuities 200

K

Khayyám, Omar 73

L

Law of Total Probability 180
Lebesgue, Henri Léon 208
l'Hôpital, Marquis de 209
L'Hôpital's Rule 209
Lines in three dimensions 149

M

Maclaurin, Colin 260
Maclaurin Series 260, 262
Magnitude of a vector 126
Mathematical Induction 49
Mean 192
Median 192
Mode 192
Modelling using Trigonometric Functions 120
Multiplication principle 2
Music 120

N

Negative vectors 127
Normal Form 170
Normal Vector form of a Plane 167

O

Odd function 85
Optimization Problem 220
Orientation 127

P

Parallel vectors 139
Partial Fraction 15, 232
Permutation 2, 3
Polar Form 33
Polynomial 41
Polynomial Graphs 66
Proof 47
Proof by Contradiction 47
Pythagorean Identity 107

Q

Quadratic Denominator 82
Quadratic Numerator 79
Quadratics 42

R

Rational Function 77
Rational Root Theorem 71
Reciprocal 104
Related Rates 216
Remainder Theorem 43, 66
Repeated Factor 70
Representing Vectors 126
Right-handed system 157
Roots of a Complex Number 39

S

San Diego Harbour 121
Scalar multiplication 131
Scalar Product 137
Scalar quantities 125
Scaling Variables 197
Separation of Variables 251
Series 259
Simultaneous linear equations , 57
Solving equations 89
Substitution method 58
Substitution Rule 235
Sum and Product of the Roots of a Quadratic Equation 72
Symmetry 119

T

Tartaglia, Niccolò 73
Three-dimensional Geometry 157
Tides 121
Translation 197

U

Unit vector 130

V

Variance 192
Vector equation 145
Vector Equation of a Plane 165
Vector operations 131
Vector Product 157
Vector quantities 125
Vertical asymptote 77
Vertical translation 226
Volumes of Revolution 242

X

Xiaotong, Wang 73

Z

Zero vector 127

# MATHEMATICS
## Analysis and Approaches - HL

IBID Press has published for the IB courses since 1998. Our books are specifically written for the courses they serve and are not adaptations from other sources.

For the new mathematics courses we have produced 5 texts which cover the Common Core, Analysis and Approaches SL & Applications and Interpretations SL and Analysis and Approaches HL & Applications and Interpretations HL. This format will help schools who need to teach as many joint classes as possible while providing the specific coverage necessary for the individual courses and the distinct levels.

Each book is in full colour, contains clear explanations, associated diagrams, cultural and international references, extensive exercises and Toolboxes.

ISBN: 978-1-921784-82-8

ISBN: 978-1-921784-82-8
9 781921 784828

For use with the I.B. Diploma Programme